No Doing Without Money: Global Indeterminacy and Equilibrium Selection in the New Keynesian Model*

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Abstract

Does money still matter for equilibrium selection as an economy becomes cashless? We adapt the cash-credit economy of Woodford (1998) to a continuous-time New Keynesian model with an exact IS equation and Phillips curve. As the cash share vanishes, real money balances and the liquidity wedge disappear, and equilibrium conditions converge to those of the cashless model. Yet an interest-rate rule satisfying the Taylor principle leaves *global multiplicity*: the fully stabilized allocation coexists with a family of stationary stochastic equilibria featuring underproduction and below-target inflation. We show that money demand, normalized by the cash share, remains nondegenerate in the limit, allowing money-supply commitments to select among these equilibria even as cash use becomes arbitrarily small. A commitment to supply money at natural output uniquely selects full stabilization, even under Taylor rules that violate the Taylor principle. An alternative inflation-indexed money supply selects the stochastic equilibrium. At the cashless end-point, however, money-market clearing becomes vacuous and money can no longer select equilibria. Thus, equilibrium conditions converge to the cashless limit, but equilibrium selection does not.

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1 Introduction

Can a monetary economy in which money plays an arbitrarily small role dispense with money for equilibrium selection? The standard cashless-limit argument suggests that it can. For example, in the cash-credit economy of Woodford (1998), an interest-rate rule, together with an appropriate fiscal policy, determines a locally unique equilibrium even as the share of transactions requiring cash approaches zero. The locally unique equilibrium varies continuously with this share, so that the cashless model provides a limiting approximation to economies in which money is used in only a negligible fraction of transactions. In the resulting New Keynesian benchmark, money demand determines the quantity of money required to implement the interest-rate rule but does not otherwise affect the equilibrium allocation (Woodford, 2008). This argument, however, is inherently local. We ask whether the cashless limit continues to hold even if the model is solved globally and the quantity of money can play a role in selecting among multiple equilibria.

We adapt the cash-credit economy of Woodford (1998) to a continuous-time New Keynesian model with stochastic productivity, retaining the exact IS equation and Phillips curve. Households can purchase goods on credit when sellers can verify their creditworthiness, but must use money for other transactions. Holding money rather than the interest-bearing short account entails an opportunity cost. This cost generates a liquidity wedge in households' marginal utility and labor supply, as well as in firms' pricing decisions. As the share of credit transactions approaches one, real money balances and the associated wedge vanish, and the equilibrium conditions converge to those of the exact cashless New Keynesian model. Thus, the cashless limit preserves the core approximation result of Woodford (1998) at the level of equilibrium conditions.

We make two contributions. First, we show that global multiplicity driven by self-fulfilling aggregate volatility survives an exact supply block,[1] even under interest-rate rules that satisfy the Taylor principle and ensure local determinacy. Second, we show that money-supply commitments can select among these global equilibria, even when the share of transactions requiring cash is arbitrarily small. The distinction is between local determinacy and global equilibrium selection. An interest-rate rule can uniquely determine the equilibrium in a neighborhood of the perfectly stabilized allocation without ruling out other global equilibria. Money supply becomes therefore irrelevant for local equilibrium determination under a Taylor rule that ensures local determinacy, but can nevertheless play a role in selecting among global equilibria.

This selection role of money, however, disappears discontinuously in the cashless end-point. Even as the share of cash transactions converges to zero, the money-market clearing condition remains operative, allowing a money supply commitment to restrict the set of global equilibria.[2] At the exact cashless end-point, by contrast, money demand and supply are both zero, so money-

[1] In our earlier paper (Lee and Dordal i Carreras, 2026), we establish multiple equilibria in a New Keynesian model with a nonlinear demand block but a log-linearized supply block.

[2] Although real money balances vanish, money demand normalized by the cash share converges to a nondegenerate limit.

market clearing imposes no restriction on equilibrium allocations. The money-supply commitment consequently loses its selection role, leaving the global multiplicity under the interest-rate rule unresolved. Thus, the cashless limit preserves the convergence of equilibrium conditions but not the equilibrium-selection mechanism: an arbitrarily small role for money can matter for global equilibrium selection, whereas zero role for money cannot.

Our model retains the precautionary-saving mechanism of Lee and Dordal i Carreras (2026) while allowing firms to set prices subject to quadratic adjustment costs à la Rotemberg (1982). Both the IS equation and the Phillips curve remain exact. Under any interest-rate rule satisfying the Taylor principle, the fully stabilized path coexists with a two-parameter family of stationary equilibria with stochastic volatility. In each member of this family, output remains below its natural level, inflation remains below target, and excess output volatility is positive almost surely. Different inflation functions generate distinct stationary distributions. Thus, global multiplicity is not an artifact of suppressing or linearizing the supply response. The exact pricing equation maps weak demand into low marginal costs and below-target inflation, while remaining consistent with the volatility process that sustains precautionary saving.

Surprisingly, the multiplicity persists when nominal interest rates are subject to a constant effective lower bound (ELB) that binds occasionally. In each constrained equilibrium, the economy spends positive stationary probabilities in both the constrained and unconstrained regimes. This is similar to the alternative zero-lower-bound equilibria driven by self-fulfilling volatility in Dordal i Carreras and Lee (2026).

A 'passive' money issuer that accommodates realized money demand on the equilibrium path leaves both the fully stabilized allocation and every stochastic member of the family as equilibria. However, although real money balances vanish as the cash share approaches zero, money demand normalized by the cash share converges to a nondegenerate limit. An 'active' commitment to this normalized quantity can therefore select among global equilibria. In particular, supplying normalized money demand at natural output uniquely picks the fully stabilized path whenever the policy rule's coefficient on inflation differs from one.[3] Alternatively, an inflation-indexed money-supply rule selects a particular stochastic equilibrium when combined with a restriction preventing real purchases from falling following positive productivity innovations. Both selection results also require an initial money stock consistent with the selected initial output gap level. At the cashless end-point, however, money demand and supply are both zero, so the money-market clearing condition imposes no restriction on the allocation. The active money-supply commitment then loses its selection role, and the global multiplicity under the interest-rate rule remains. Therefore, while the equilibrium conditions converge to those of the cashless model, the global equilibrium-selection mechanism need not survive the limit.

Thus, the distinction between arbitrarily vanishing share of cash trasaction and an exactly zero cash usage is central to our results. For every positive cash share, a money supply commitment

[3]It implies that the Taylor principle is not required for this selection result.

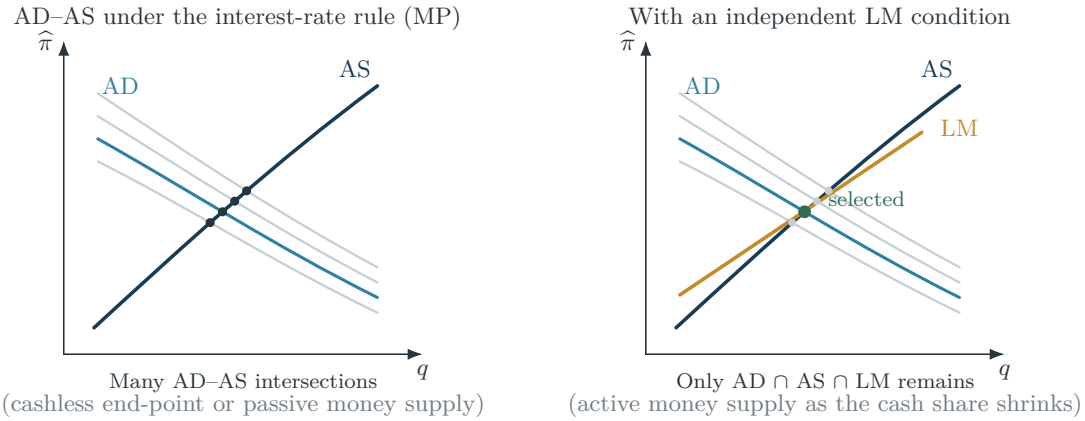


Figure 1: A textbook metaphor for cash-market selection. Under the interest-rate rule, aggregate demand is the IS–MP schedule and aggregate supply is the Phillips curve. Self-fulfilling excess volatility shifts AD, so AD and AS can intersect at many points. When money supply is passive, or at the cashless end-point, the LM condition is residual and does not cut this set. An active money supply restores an independent LM schedule and selects the unique intersection of AD, AS and LM. The figure is stylized.[5]

can select an allocation, even as the associated real money balances become arbitrarily small. At the cashless end-point, however, nobody holds money, and money-market clearing imposes no restriction on the allocation. The interest-rate rule alone admits both the fully stabilized path and the stationary stochastic family, without selecting among them. Figure 1 restates the same logic in textbook AD–AS language.

Literature. The first contribution builds on our two earlier papers. Lee and Dordal i Carreras (2026) show that conventional Taylor rules deliver local determinacy but leave multiple bounded global equilibria. Households that anticipate greater future consumption volatility save more for precautionary reasons. Current demand and output fall, and state-contingent plans for future consumption validate the volatility that households themselves anticipated.[6] Dordal i Carreras and Lee (2026) study aggregate volatility at the zero lower bound (ZLB) in a rigid-price economy. By relinquishing its commitment to perfect stabilization post the ZLB, the central bank can use higher-order forward guidance to coordinate households on a more favorable equilibrium path, characterized by lower aggregate volatility and a smaller precautionary risk premium during the ZLB. This improves the output gap and raises welfare relative to conventional forward guidance,

[5] See Jones (2020) and Romer (2019) for the IS–MP representation of aggregate demand (AD). In the textbook IS–LM diagram, the LM schedule is drawn in (Y, i) space. Here we plot an analogous money-market restriction in $(q, \widehat{\pi})$ space only as a metaphor. Section 3 microfounds the relevant money-market clearing condition in the cashless limit, where normalized balances retain a nondegenerate demand and an active money supply can select among the AD–AS intersections.

[6] Lee and Dordal i Carreras (2026) builds analytical constructions based on a log-linearized Phillips curve, while the fully non-linear pricing block is treated numerically.

although it entails a trade-off with stabilization after the guidance period.[7]

A large literature studies the determinacy of the New Keynesian model under Taylor rules. Local analysis typically finds that sufficiently strong responses to inflation or economic activity ensure a unique bounded equilibrium around the non-stochastic steady state (Clarida et al., 1999; Woodford, 2003; Galí, 2015). This conclusion, however, rests on a local rational-expectations analysis that rules out explosive paths (Blanchard and Kahn, 1980).[8] Related work establishes the existence of global equilibria that lie outside the neighborhood considered by local determinacy analysis. In particular, Benhabib et al. (2002) construct liquidity-trap equilibria under active Taylor rules, while both Acharya and Dogra (2020) and Acharya and Benhabib (2026) show that countercyclical idiosyncratic risk can generate indeterminacy in heterogeneous-agent economies, even under aggressive inflation responses. Khorrami and Mendo (2026) study uncertainty-driven multiplicity and fiscal equilibrium selection, retaining a precautionary term in aggregate demand while relying primarily on a linearized supply block. Our analysis identifies a distinct source of global multiplicity: self-fulfilling aggregate volatility in a representative-agent economy. We show analytically that this mechanism survives an exact non-linear pricing equation, and that money-supply commitments can select among global equilibria even as the share of cash transactions approaches zero.

Our money-based selection mechanism relates to work that uses monetary commitments to eliminate equilibria left open by interest-rate rules (Christiano and Rostagno, 2001; Atkeson et al., 2010). Its implication for the cashless limit, however, is very distinct. Although the money supply commitment converges to zero in levels, its normalization continues to impose a non-degenerate restriction on equilibrium for every small positive cash share. Lagos (2026) also revisits the role of money in cashless limits. With costly credit intermediation, the limiting monetary allocation in Lagos (2026) can differ from the allocation of the corresponding nonmonetary economy. In our setting, by contrast, the equilibrium conditions converge to those of the cashless economy. What does not carry over to the limit is thae equilibrium selection mechanism: at a zero cash share, the money-market clearing condition becomes vacuous, and the multiplicity left by the interest-rate rule remains unresolved.

The selection mechanism also has a structural analogy to the fiscal theory of the price level. Under an interest-rate rule, an active fiscal authority does not adjust future surpluses to validate every candidate price-level path. The government valuation equation then restricts equilibrium and can determine the price level (Leeper, 1991; Sims, 1994; Woodford, 1995; Cochrane, 2023). Our normalized money-supply rules play a parallel role: they do not accommodate money demand

[7] In Dordal i Carreras and Lee (2026), perfect stabilization at a future date eliminates current excess volatility at the ZLB and selects a unique equilibrium. In the stationary equilibrium class constructed here with the effective lower bound, every future date repeats the same policy: the nominal interest rate follows the Taylor rule or is at the ELB, depending on the current state. Under that recurrence, stationary equilibria persist, and excess volatility remains positive almost surely.

[8] Cochrane (2011) questions whether private-sector optimality, e.g., transversality condition, alone provides a sufficient basis for excluding such paths.

at every candidate allocation, so money-market clearing becomes an equilibrium restriction rather than a residual equation for the money stock. In this regime sense, the selecting money-supply rules are active, whereas the rule that accommodates realized money demand is passive.[9]

Layout Section 2 presents the cash-credit economy, individual choices, and monetary policy. Section 3 derives the cashless limit, establishes global multiplicity under passive money supply, and presents the two selecting money-supply rules. Section 4 concludes. Appendices A–D derive household and producer optimality conditions and the local and flexible-price benchmarks.

Online Appendix A constructs the stochastic equilibrium family, Online Appendix B verifies global individual optimality for households and firms, and Online Appendix C proves the selection results of Section 3.

2 The cash-credit economy

2.1 Locations, goods and preferences

Time is continuous,[10] with one unit of time normalized to a quarter. There is a unit mass of households and a unit mass of sticky-price producers. Following Woodford (1998), we assume that households live on a circle of circumference one and are indexed by their location h. Consumption also takes place at locations k on the same circle, and every household consumes at every location. We write $|k-h|$ for the arc distance between two locations.

Producers are indexed by $j \in [0,1]$. Each makes a differentiated variety and sells it at its own price p_t^j at every location. A household's consumption at location k is the standard Dixit–Stiglitz basket of the varieties it buys there, with elasticity $\varepsilon > 1$:

$$c_t^h(k) = \left[\int_0^1 (x_t^{h,k,j})^{(\varepsilon-1)/\varepsilon} dj\right]^{\varepsilon/(\varepsilon-1)},$$

where $x_t^{h,k,j}$ is the quantity of variety j that household h purchases at location k. Note that every location offers the same varieties at the same prices. Locations differ only in where consumption takes place and, as described below, in how the household can pay.

Buying a basket at minimum cost gives the price index p_t and the variety demands $x_t^{h,k,j}$. Producer j meets all demand for its variety at its posted price, so its output sums these demands

[9] Note, however, that the underlying channels are different. Our issuer has no net nominal government debt, and no fiscal solvency or government-wealth condition selects prices. Selection operates through transaction demand and a quantity commitment, and it disappears at the literally cashless end-point.

[10] As shown in Lee and Dordal i Carreras (2026), the choice of continuous versus discrete time is immaterial for our results: all of our findings continue to hold in a discrete-time setting under regularity conditions.

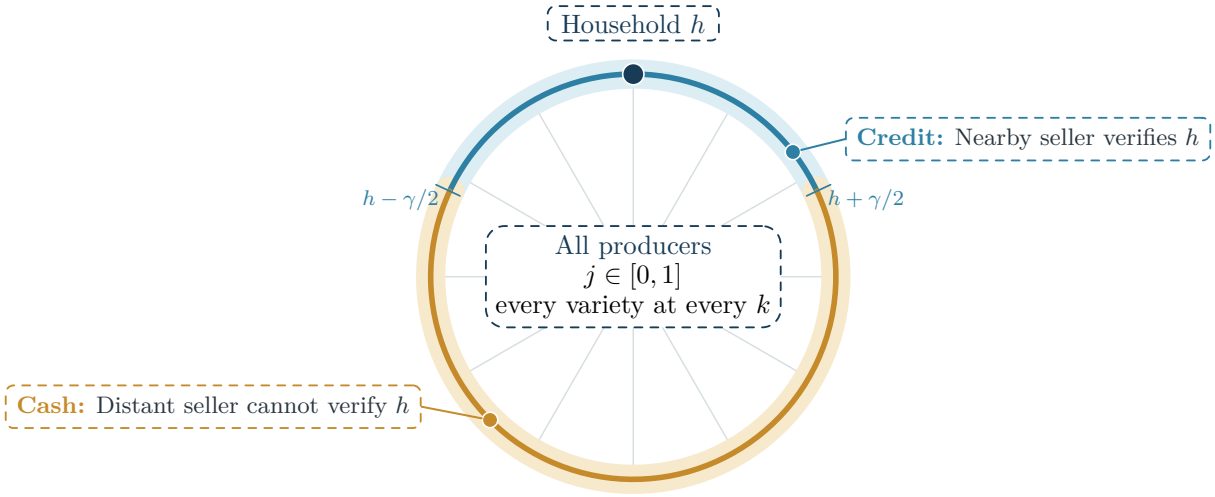


Figure 2: The circular cash–credit economy adapted from Woodford (1998). Household h can buy on credit at locations within arc distance $\gamma/2$ and must use money at more distant locations. Every producer sells its variety at every location.

over locations and households:

$$
\begin{aligned}
p_t &= \left[\int_0^1 (p_t^j)^{1-\varepsilon} dj\right]^{1/(1-\varepsilon)}, & x_t^{h,k,j} &= \left(\frac{p_t^j}{p_t}\right)^{-\varepsilon} c_t^h(k), \\
C_t^h &= \int_0^1 c_t^h(k) dk, & C_t &= \int_0^1 C_t^h dh, \\
Y_t^j &= \int_0^1 \int_0^1 x_t^{h,k,j} dk\, dh = \left(\frac{p_t^j}{p_t}\right)^{-\varepsilon} C_t.
\end{aligned}
\tag{2.1}
$$

The price index p_t is the minimum cost of a basket at every location. Producer j's real output is Y_t^j, household h's real spending on goods is C_t^h, and aggregate real resource consumption is C_t. Neither the index nor producer demand depends on how households divide their purchases between cash and credit, since every location sells the same basket. Effectively, giving all locations the same varieties keeps the two frictions separate. Payment frictions act on where households buy, and nominal price rigidity acts on producers.

Payment depends on distance. A household can buy on credit at locations within arc distance $\gamma/2$ of its own, where sellers can verify its creditworthiness, but must pay with money farther away. Its credit and cash locations are therefore

$$
\mathcal{K}_{\text{credit}}^h = \{k : |k-h| \le \gamma/2\}, \qquad \mathcal{K}_{\text{cash}}^h = \{k : |k-h| > \gamma/2\},
$$

with measures γ and $1-\gamma$. We take $\gamma \in (0,1)$ and study the cashless limit $\gamma \to 1$ in Section 3.

The household's cash and credit purchases are

$$X_t^h = \int_{\mathcal{K}_{\text{cash}}^h} c_t^h(k)dk, \qquad V_t^h = \int_{\mathcal{K}_{\text{credit}}^h} c_t^h(k)dk, \qquad C_t^h = X_t^h + V_t^h.$$

Thus, whether a purchase requires money thus depends on the buyer and the location, not on the varieties bought. Each household can use credit at a share γ of locations, and each location sells on credit to a share γ of households.[11] Households choose how to split their spending between cash and credit.

Households regard consumption at different locations as imperfect substitutes, with elasticity $\theta > 1$. With labor L_t^h and Frisch elasticity $\eta > 0$, flow utility is

$$\mathcal{C}_t^h = \left[\int_0^1 c_t^h(k)^{1-1/\theta}dk\right]^{\theta/(\theta-1)}, \qquad U_t^h = \log \mathcal{C}_t^h - \frac{(L_t^h)^{1+1/\eta}}{1+1/\eta}. \tag{2.2}$$

Households maximize $\mathbb{E}_t \int_t^\infty e^{-\rho(s-t)} U_s^h ds$, with discount rate $\rho > 0$. Because the index is concave, households prefer to spread consumption across locations. Without money they could buy only at credit locations near home. Money makes distant purchases possible, but because holding it is costly, households still buy less at each cash location than at each credit location.

Finally, productivity is common to all producers, and inflation is the drift of the price index:

$$\frac{dA_t}{A_t} = g\,dt + \sigma\,dZ_t, \qquad \frac{dp_t}{p_t} = \pi_t dt, \qquad A_0, p_0 > 0, \quad \sigma > 0. \tag{2.3}$$

Here Z_t is a standard Brownian motion with respect to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ that satisfies the usual conditions, and the initial levels A_0 and p_0 are known at time zero.

2.2 Payments, money and budgets

Households can pay in two ways (credit and cash), and sellers are paid immediately under both. Credit purchases are charged to the household's short account, which pays the nominal interest rate i_t on positive balances and charges the same rate on negative balances, like a card drawn on an overdraft line at the market rate i_t. A credit purchase carries no interest charge of its own. It lowers the balance by the price, and the balance continues to earn or pay i_t. Accepting credit requires trust in the buyer's solvency, which only nearby sellers have.

By contrast, money is an asset used primarily for settlement and does not require such trust. It is issued by an issuer that lends it to households and pays a nominal interest rate i_t^m.[12] Money is mapped to currency, a low-remuneration deposit, or a digital settlement balance. Importantly, it must be held in advance of a cash purchase and earns no higher nominal return than i_t.

[11] Note that the parameter γ is a share of locations, not of spending.
[12] The literature typically assumes $i_t^m = 0$ for notational convenience.

At a household's cash locations $\mathcal{K}^h_{\text{cash}}$, sellers accept only money. For each cash location, the household sends a continuum of shoppers indexed by $\iota \in [0,1]$, who receive unanticipated Poisson purchase opportunities at rate $v > 0$. Wallets can be replenished between opportunities but not between an opportunity's arrival and the purchase, so every cash purchase must be funded in advance. Sellers can convert money receipts into account balances at par, so accepting money imposes no working-capital cost on them.

Let M^h_t be the household's nominal money and $m^h_t = M^h_t/p_t$ its real balances. Let $m^h_t(k,\iota)$ be the part of these balances carried in a shopper's wallet and $\chi^h_t(k,\iota)$ the quantity it is instructed to buy at an opportunity. Both are predictable, with $0 \le \chi^h_t(k,\iota) \le m^h_t(k,\iota)$, and total wallet balances cannot exceed m^h_t. Due to a law of large numbers across shoppers, at each location the realized purchase flow equals its expected value, v times the integral of the instructions. Hence

$$c^h_t(k) = v\int_0^1 \chi^h_t(k,\iota)d\iota,$$
$$X^h_t = v\int_{\mathcal{K}^h_{\text{cash}}}\int_0^1 \chi^h_t(k,\iota)d\iota\,dk \le v\int_{\mathcal{K}^h_{\text{cash}}}\int_0^1 m^h_t(k,\iota)d\iota\,dk \le vm^h_t.$$

The household therefore faces the cash constraint

$$X^h_t \le vm^h_t. \tag{2.4}$$

Conversely, the household can implement any nonnegative, integrable profile of purchases $c^h_t(k)$ at cash locations that is chosen in advance and satisfies (2.4). It suffices that each shopper at a cash location carries $m^h_t(k,\iota) = c^h_t(k)/v$ and spends all of it at its next opportunity, $\chi^h_t(k,\iota) = m^h_t(k,\iota)$, while the household keeps any balances beyond X^h_t/v outside the wallets. The cash constraint is therefore necessary and sufficient. Intuitively, a shopper waits on average $1/v$ units of time for an opportunity, so a flow of cash spending X^h_t requires real balances of at least X^h_t/v, and v is the maximum velocity of cash purchases.

Sellers charge the same price whether they are paid with money or on credit. Money, however, must be held before it is spent, and in the meantime it earns i^m_t rather than the short-account rate i_t. Its carrying cost is the spread $\delta_t = i_t - i^m_t$, which is nonnegative because the issuer never lends below the rate it pays on money. A unit of consumption paid with money thus also requires $1/v$ units of real balances held in advance at spread δ_t. Measured in units of consumption bought on credit, its effective cost is

$$R_t = 1 + \delta_t/v \ge 1. \tag{2.5}$$

We call R_t the relative cost of cash consumption; the corresponding nominal cost is p_tR_t.[13] With

[13] Woodford (1998) derives a similar relative cost of cash transactions from the seller's perspective: sellers impose an interest charge on cash purchases to compensate for the delay in receiving payment. By contrast, sellers charge the same price regardless of the payment method in our setting, and the wedge arises from the buyer's opportunity cost of holding money in advance.

a positive spread, $R_t > 1$ and idle balances are costly, so households pay with money only where they must, at cash locations, and hold no more of it than those purchases require, implying that the cash constraint (2.4) must bind with $m_t^h = X_t^h/v$. With a zero spread, $R_t = 1$ and households are indifferent between the two instruments and between money and the short account as stores of value. We break this tie by assuming that they then pay with credit wherever it is accepted and hold no balances beyond those required for cash purchases, so that the cash constraint still binds and money demand remains determinate at all times.

Given these costs, the household attains its consumption basket $\mathcal{C}_t^h$ in (2.2) at minimum cost by buying at each location according to:

$$c_t^h(k) = \left(\frac{p_t R_t}{P_{\gamma,t}}\right)^{-\theta} \mathcal{C}_t^h, \quad k \in \mathcal{K}_{\text{cash}}^h, \qquad\qquad c_t^h(k) = \left(\frac{p_t}{P_{\gamma,t}}\right)^{-\theta} \mathcal{C}_t^h, \quad k \in \mathcal{K}_{\text{credit}}^h, \tag{2.6}$$

where

$$P_{\gamma,t} = p_t[\gamma + (1-\gamma)R_t^{1-\theta}]^{1/(1-\theta)} \tag{2.7}$$

is the effective price index of the consumption basket. It aggregates the nominal costs $p_t R_t$ at cash locations and p_t at credit locations, so $P_{\gamma,t}\mathcal{C}_t^h$ is the minimum nominal cost of the basket. The household thus spreads its purchases evenly within each payment class, buying $X_t^h/(1-\gamma)$ at each cash location and V_t^h/γ at each credit location. Together with $C_t^h = X_t^h + V_t^h$ and the binding cash constraint (2.4), this allocation determines the household's real money demand, its optimal real balances, and its purchases by payment clas,[14]

$$m_t^{h,\mathrm{d}} = d_{\gamma,t} C_t^h, \qquad X_t^h = v d_{\gamma,t} C_t^h, \qquad V_t^h = [1 - v d_{\gamma,t}] C_t^h, \qquad d_{\gamma,t} = \frac{1-\gamma}{v[1-\gamma+\gamma R_t^\theta]}, \tag{2.8}$$

where $d_{\gamma,t}$ is real money demanded per unit of resource consumption. At the optimum $m_t^h = m_t^{h,\mathrm{d}}$, so that each unit of resource consumption carries a money cost of $\delta_t d_{\gamma,t}$. Adding this carrying cost to resource consumption gives *real expenditure*,

$$Q_t^h = \frac{P_{\gamma,t}\mathcal{C}_t^h}{p_t} = C_t^h + \delta_t m_t^h = C_t^h H_{\gamma,t}, \qquad H_{\gamma,t} = 1 + \delta_t d_{\gamma,t}, \tag{2.9}$$

where $H_{\gamma,t}$ is the *liquidity wedge*, or equivalently, the average relative cost per unit of resource consumption. We write its log dynamics as

$$d\log H_{\gamma,t} = g_{H,t}dt + \sigma_t^H dZ_t. \tag{2.10}$$

The household's only financial assets are money and its short-account position B_t^h, which can be negative in the borrowing case. Define real wealth $a_t^h = (B_t^h + M_t^h)/p_t$ and the real rate $r_t = i_t - \pi_t$. Wealth earns the real rate and falls with real expenditure, because a purchase costs

[14] These are derived in Appendix A.2.

the same whether it is paid with money or on credit and holding money adds only its carrying cost. The dynamic budget constraint is therefore[15]

$$da_t^h = \left[r_t a_t^h + \frac{w_t}{p_t} L_t^h + \frac{D_t}{p_t} + \mathcal{T}_t - Q_t^h \right] dt. \tag{2.11}$$

Here w_t is the nominal wage, D_t collects nominal producer profits and fee rebates, and $\mathcal{T}_t$ is the issuer's real transfer, all taken as given by the household.

2.3 The household problem

Household h chooses its consumption basket $\mathcal{C}_t^h$, labor L_t^h and wealth a_t^h to maximize expected discounted utility (2.2) subject to the budget flow (2.11), taking prices and transfers as given. Its purchases at each location and its money balances follow from Section 2.2.[16]

The first-order condition for the basket sets the marginal value of real wealth equal to $1/Q_t^h$, the inverse of real expenditure. Combined with the first-order condition for labor, it gives the intratemporal condition

$$\frac{1}{Q_t^h} \frac{w_t}{p_t} = (L_t^h)^{1/\eta}, \tag{2.12}$$

which equates the real wage, valued at this marginal value, with the marginal disutility of labor.

Let $\xi_t^h = e^{-\rho t}/Q_t^h$ be the household's effective stochastic discount factor and $\xi_t^{N,h} = \xi_t^h / p_t$ its nominal counterpart.[17] At an interior optimum, the short-account Euler equation is

$$\mathbb{E}_t[d\xi_t^{N,h}] = -i_t \xi_t^{N,h} dt. \tag{2.13}$$

Natural output and the IS equation. For the aggregate equations, consider continuous Itô equilibria adapted to the augmented filtration generated by Z, so all aggregate innovations load on the productivity shock. Symmetric goods-market clearing[18] and linear production give $C_t = Y_t = A_t L_t$. A flexible-price producer charges the markup $1/\alpha$, where $\alpha = (\varepsilon - 1)/\varepsilon$. Natural output and labor are those of flexible-price producers at current productivity and the spread δ_t (Woodford, 2003; Galí, 2015)[19]:

$$Y_t^n = A_t L_t^n, \qquad L_t^n = \left(\frac{\alpha}{H_{\gamma,t}} \right)^{\eta/(1+\eta)}, \qquad q_t = \frac{Y_t}{Y_t^n}, \tag{2.14}$$

[15] The general budget constraint, which holds for arbitrary purchases and balances, charges resource consumption C_t^h and the carrying cost $\delta_t m_t^h$ separately, which reduces to (2.11) from (2.9). See Appendix A.1 and Appendix A.2.

[16] Appendix A gives the detailed derivations. For global optimality and sufficient conditions, Online Appendix B.1 presents the sufficient condition, and Online Appendix B.3 verifies the sufficient condition for equilibria of Section 3 under cashless limit.

[17] Note that the effectiver discount factor $\xi_t^{N,h}$ is written in Q_t^h, not $\mathcal{C}_t^h$ or C_t^h.

[18] It will be formally stated in Section 2.6.

[19] Appendix A.3 derives the flexible-price economy's output and labor in (2.14), and the natural rate of interest (2.15a).

where $q_t - 1$ is the output gap. Note that the flexible-price benchmark retains the cash frictions and production markup: in (2.14), L_t^n conditions on actual payment terms $H_{\gamma,t}$ that depends on the current spread, δ_t.[20]

Applying Itô's formula to the household's Euler condition (2.13) gives the natural rate and the exact IS equation:

$$r_t^n = \rho + g + \frac{g_{H,t}}{1+\eta} - \frac{\sigma^2 + (\sigma_t^n)^2}{2}, \tag{2.15a}$$

$$\frac{dq_t}{q_t} = [i_t - \pi_t - r_t^n + \sigma_t^s(\sigma_t^n + \sigma_t^s)]\,dt + \sigma_t^s dZ_t. \tag{2.15b}$$

where $\sigma_t^n = \sigma + \sigma_t^H/(1+\eta)$ is the natural volatility of real expenditure and σ_t^s its excess volatility under sticky prices, so that real expenditure Q_t has total volatility $\sigma_t^n + \sigma_t^s$. Note that (2.15b) is similar to the IS equation in Lee and Dordal i Carreras (2026) except that now it is written in q_t, not $\log q_t$.

2.4 Producers and price setting

Producer j hires labor at the common wage, produces with the linear technology $Y_t^j = A_t L_t^j$ and faces the demand curve in (2.1). Its price moves continuously, $dp_t^j/p_t^j = \pi_t^j dt$, and the producer chooses its own inflation rate π_t^j. Price changes that depart from a fixed indexation rate $\bar{\pi}$ are costly. The firm pays a quadratic price-adjustment fee (Rotemberg, 1982),

$$\Theta_t^j = \frac{\tau}{2}(\pi_t^j - \bar{\pi})^2 p_t^j Y_t^j, \qquad \tau > 0, \tag{2.16}$$

which is proportional to its nominal revenues and to the squared gap between its own inflation π_t^j and the target rate $\bar{\pi}$. The parameter τ scales the cost of adjustment, and there is no cost when $\pi_t^j = \bar{\pi}$. We assume that the fees (2.16) are lump-sum rebated to households, independently of any individual choice, to simplify market clearing.[21] Nominal profit is revenue net of the wage bill and the fee,

$$\Psi_t^j = p_t^j Y_t^j - w_t L_t^j - \Theta_t^j. \tag{2.17}$$

Households are identical, so in a symmetric equilibrium they share the stochastic discount factor $\xi_t = e^{-\rho t}/Q_t$, which producers use to value the profit stream. From any date t, producer j chooses its inflation path $\{\pi_s^j\}_{s \geq t}$ to maximize the expected present value of its real profits,

$$\mathbb{E}_t \int_t^\infty \frac{\xi_s}{\xi_t} \frac{\Psi_s^j}{p_s} ds,$$

[20] Eventually, as the cash share vanishes ($\gamma \to 1$), Section 3 shows that the equilibrium conditions converge to their cashless limit, in which the endogenous term $H_{\gamma,t}$ and its drift and volatility, $g_{H,t}$ and σ_t^H, no longer appear.

[21] The fees are then transfers rather than a use of output, so goods-market clearing equates consumption with output.

taking the wage, the price index and aggregate quantities as given. Appendix B derives producer j's first-order conditions.[22]

The New Keynesian Phillips curve. In a symmetric equilibrium all producers j choose the same inflation rate $\pi_t^j = \pi_t$. With $\widehat{\pi}_t = \pi_t - \bar{\pi}$ being inflation in excess of indexation, let σ_t^π be its loading on Z, and write $\kappa = (\varepsilon - 1)/\tau$. With natural output (2.14), the producers' first-order conditions give the New Keynesian Phillips curve given by:

$$d\widehat{\pi}_t = \left[\left(\rho + g_{H,t} - \frac{(\sigma_t^H)^2}{2} \right) \widehat{\pi}_t - \kappa \left(q_t^{1+1/\eta} - 1 \right) - \frac{\varepsilon - 1}{2} \widehat{\pi}_t^2 + \sigma_t^H \sigma_t^\pi \right] dt + \sigma_t^\pi dZ_t. \tag{2.18}$$

The output ratio q_t in (2.14) determines the marginal-cost gap, while changes in the liquidity wedge $H_{\gamma,t}$ affect inflation's drift through its growth, variance and covariance with inflation. If the wedge is absolutely continuous, the two terms involving σ_t^H vanish. The remaining quadratic inflation term comes from the own-sales adjustment fee. Equation (2.18) is exact rather than log-linearized; Appendix B.2 derives it directly from the individual pricing conditions.

2.5 The money issuer and monetary policy

The issuer is the monetary authority, such as the central bank. It creates money by lending to households and sets the short-term nominal interest rate i_t at which it lends, the interest rate i_t^m it pays on money, and the money stock. Its only assets are loans, $K_t \geq 0$, and its only liabilities are the money stock, M_t^{s}. Thus, it does not borrow from households. Loans earn i_t, money pays i_t^m, and the issuer rebates the margin back to households as the nominal transfer $p_t \mathcal{T}_t$. Hence

$$dK_t = [i_t K_t - i_t^m M_t^{\mathrm{s}} - p_t \mathcal{T}_t] dt + dM_t^{\mathrm{s}}, \qquad p_t \mathcal{T}_t = \delta_t M_t^{\mathrm{s}}. \tag{2.19}$$

The issuer's net worth $K_t - M_t^{\mathrm{s}}$ therefore evolves according to $d(K_t - M_t^{\mathrm{s}}) = i_t (K_t - M_t^{\mathrm{s}}) dt$. We assume zero initial net worth, $K_0 = M_0^{\mathrm{s}}$, so net worth remains zero and loans equal the money stock at all dates. With no capital, no income other than loan interest and no power to tax, the issuer cannot pay more on money than its loans earn. It therefore never lends below its cost of funds, the rate i_t^m it pays on money, so $i_t \geq i_t^m$ at all dates and the transfer is nonnegative.

Subject to $i_t \geq i_t^m$, the issuer sets both rates freely. The conditions of Sections 2.3 and 2.4 hold for any nonnegative spread path, so the remuneration of money need not be specified further. The policy rate follows the Taylor rule

$$i_t = r^n + \bar{\pi} + \phi_y (q_t - 1) + \phi_\pi (\pi_t - \bar{\pi}), \qquad r^n = \rho + g - \sigma^2, \tag{2.20}$$

[22] Online Appendix B.2 derives the admissible inflation paths and a sufficient condition for global optimality, and Online Appendix B.3 verifies it for the limiting equilibria.

where the inflation target equals the indexation rate $\bar{\pi}$, and ϕ_y and ϕ_π set the responses to the output gap and to inflation's deviation from target. The intercept r^n is the natural rate (2.15a) when the liquidity wedge is constant, as at a constant spread, so that $g_{H,t} = \sigma_t^H = 0$. The money stock is the issuer's remaining instrument. Because prices are sticky, the price level cannot jump, so the issuer controls real money supply $m_t^{\mathrm{s}} = M_t^{\mathrm{s}}/p_t$ through the nominal stock.

2.6 Market clearing and equilibrium

We study symmetric equilibria, in which households choose identical plans and producers charge identical prices, so $p_t^j = p_t$ and $Y_t^j = Y_t$. Goods, labor, money and the short account then clear when

$$C_t = Y_t = A_t L_t, \qquad m_t^{\mathrm{s}} = m_t^{\mathrm{d}} = d_{\gamma,t} Y_t, \qquad B_t^h = -K_t. \tag{2.21}$$

Because the adjustment costs (2.16) are rebated, goods clearing equates consumption with output. Money clearing equates real money supply with aggregate real money demand $m_t^{\mathrm{d}} = \int_0^1 m_t^{h,\mathrm{d}} dh$, determined by (2.8) and goods clearing.

Without a superscript, let M_t and $m_t = M_t/p_t$ denote equilibrium money and real balances. The momeny market clearing condition gives the following quantity equation

$$M_t \mathcal{V}_{\gamma,t} = p_t Y_t, \tag{2.22}$$

where the aggregate velocity of money is $\mathcal{V}_{\gamma,t} = 1/d_{\gamma,t}$. From (2.8), $\mathcal{V}_{\gamma,t}$ equals the velocity of cash purchases, v, divided by the share of purchases paid in cash:

$$\mathcal{V}_{\gamma,t} = v \times \frac{Y_t}{X_t}.$$

Real balances times velocity therefore equal real purchases, $m_t \mathcal{V}_{\gamma,t} = Y_t$, the sum of cash and credit purchases at producer prices. The short account, where households finance these balances, clears because the issuer is the only lender, so each household's balance is its loan from the issuer. Since $K_t = M_t^{\mathrm{s}}$, household net wealth a_t^h is zero. Households borrow only to hold money, and the transfer returns their carrying cost $\delta_t m_t^h$ as a lump sum.

Definition 1 (Symmetric Equilibrium)**.** Given $\gamma \in (0,1)$, A_0, p_0, an initial money stock $M_0 > 0$, a nonnegative spread process and a money-supply rule, an equilibrium consists of progressively measurable processes $(q_t, \pi_t, i_t, \sigma_t^s, \sigma_t^\pi)$, with $q_t > 0$ and $\widehat{\pi}_t$ continuous, such that the rule (2.20), the IS equation (2.15b), the Phillips curve (2.18) and money-market clearing in (2.21) hold at all dates. Here $p_t = p_0 \exp(\int_0^t \pi_s ds)$, and the real supply m_t^{s} is given by the rule.

3 The cashless limit

Following Woodford (1998), we now take the cashless limit ($\gamma \to 1$) of the equilibrium conditions. As $d_{\gamma,t} \to 0$, real balances vanish in this limit, so we measure them per unit of cash share and mark them with a tilde, as in $\tilde{m}_t^{\mathrm{d}} = m_t^{\mathrm{d}}/(1-\gamma)$. We then obtain in the limit:

$$H_{\gamma,t} \to 1, \qquad \frac{P_{\gamma,t}}{p_t} \to 1, \qquad \tilde{d}_t = \frac{d_{\gamma,t}}{1-\gamma} \to \frac{1}{vR_t^{\theta}}. \tag{3.1}$$

The same cash share controls the dynamics of the liquidity wedge. Substituting (2.8) into (2.9) writes the wedge as a smooth function of the current spread,

$$H_{\gamma,t} = 1 + (1-\gamma)\frac{\delta_t}{v[1-\gamma+\gamma R_t^{\theta}]}.$$

For $\gamma \geq 1/2$ and every $\delta_t \geq 0$, the fraction and its first two derivatives with respect to δ_t remain bounded by constants that depend only on v and θ. The cash share $1-\gamma$ therefore factors out of both the level of the wedge and its local response to the spread. Because $\log H_{\gamma,t}$ is the logarithm of one plus a term of order $1-\gamma$, the same factor passes to the logarithm and to its first two derivatives. There is a constant Λ, depending only on v and θ, such that

$$|\log H_{\gamma,t}| + |\partial_\delta \log H_{\gamma,t}| + |\partial_{\delta\delta} \log H_{\gamma,t}| \leq \Lambda(1-\gamma),$$

uniformly in the spread.

Now let the spread δ_t itself be an Itô process, $d\delta_t = b_t^{\delta}dt + \sigma_t^{\delta}dZ_t$. Itô's formula applied to $\log H_{\gamma}(\delta_t)$ identifies the coefficients in (2.10):

$$\begin{aligned} \sigma_t^H &= \partial_\delta \log H_{\gamma,t}\, \sigma_t^{\delta}, \\ g_{H,t} &= \partial_\delta \log H_{\gamma,t}\, b_t^{\delta} + \frac{1}{2}\partial_{\delta\delta} \log H_{\gamma,t}\, (\sigma_t^{\delta})^2. \end{aligned}$$

The diffusion coefficient of the wedge σ_t^H is the spread's diffusion coefficient scaled by the slope of $\log H_{\gamma}$. The drift $g_{H,t}$ contains the usual second-derivative correction from the spread's quadratic variation. The bounds above therefore give

$$|\sigma_t^H| \leq \Lambda(1-\gamma)|\sigma_t^{\delta}|, \qquad |g_{H,t}| \leq \Lambda(1-\gamma)\big[|b_t^{\delta}| + (\sigma_t^{\delta})^2\big]. \tag{3.2}$$

The inequality (3.2) holds for every spread process. Thus, both $g_{H,t}$ and σ_t^H vanish as $\gamma \to 1$. Real expenditure $Q_t = C_t H_{\gamma,t}$ then converges to consumption. Its natural volatility $\sigma_t^n = \sigma + \sigma_t^H/(1+\eta)$ converges to σ, and the natural rate (2.15a) converges to the intercept $r^n = \rho + g - \sigma^2$ of the policy rule (2.20). The spread remains in the limiting system through $\tilde{d}_t$, which scales real balances in the money-supply rules below. With the policy rule (2.20) substituted in, the term $r^n - r_t^n$ in

the IS drift vanishes, and the IS equation (2.15b), the Phillips curve (2.18) and money-market clearing (2.21) converge to

$$\frac{dq_t}{q_t} = [\phi_y(q_t - 1) + (\phi_\pi - 1)\widehat{\pi}_t + \sigma_t^s(\sigma + \sigma_t^s)]\,dt + \sigma_t^s dZ_t, \tag{3.3a}$$

$$d\widehat{\pi}_t = \left[\rho\widehat{\pi}_t - \kappa\left(q_t^{1+1/\eta} - 1\right) - \frac{\varepsilon - 1}{2}\widehat{\pi}_t^2\right]dt + \sigma_t^\pi dZ_t, \tag{3.3b}$$

$$\tilde{m}_t^{\mathrm{s}} = \tilde{d}_t Y_t^n q_t, \tag{3.3c}$$

where the first two are the standard New Keynesian IS equation and Phillips curve in their exact non-linear form.

Definition 2 (Limiting equilibrium)**.** A limiting equilibrium is an equilibrium in Definition 1, in which money is measured per unit of cash share, with initial balances $\tilde{m}_0 > 0$, and (3.3) replace the IS equation, the New Keynesian Phillips curve and money-market clearing.

We next study the limiting equilibrium. First, Section 3.1 considers the passive money supply usually assumed in the literature, as well as the cashless economy itself ($\gamma = 1$). In both cases, the economy is indeterminate and presents two classes of solutions: a fully stabilized solution, which local analysis around the steady state selects, and a family of stationary stochastic solutions. Sections 3.2 and 3.3 then study how alternative money-supply rules can select either the fully stabilized solution or a single stochastic solution as the unique limiting equilibrium.

3.1 Passive money supply

Under a passive money supply, the issuer meets real money demand at realized output at every date, including date zero,

$$\tilde{m}_t^{\mathrm{s}} = \tilde{d}_t Y_t. \tag{3.4}$$

Money-market clearing then holds for any allocation and only determines the money stock, as in Woodford (2008). Equilibrium is therefore characterized by the IS equation (3.3a) and the Phillips curve (3.3b) alone.[23] As recent work shows, equations of this type have many solutions, including several classes of bounded global equilibria sustained by self-fulfilling volatility, even when the policy rule satisfies the Taylor principle (Lee and Dordal i Carreras, 2026; Khorrami and Mendo, 2026).

Rather than classify them all, we present two classes that suffice for our argument: a fully stabilized solution and a family of stationary stochastic solutions.

Lemma 1 (Fully stabilized solution)**.** *For any policy coefficients, the IS equation* (3.3a) *and the Phillips curve* (3.3b) *admit the solution* $q_t = 1$ *and* $\pi_t = \bar{\pi}$ *at all dates, with* $\sigma_t^s = \sigma_t^\pi = 0$.

[23] Note that the IS equation (3.3a) already includes the Taylor rule (2.20).

Proof. At $q_t = 1$ and $\widehat{\pi}_t = 0$ with zero loadings, both drifts in (3.3a) and (3.3b) vanish. □

This solution corresponds to the fully stabilized allocation, where output equals its flexible-price level at every date and state, so the output gap is zero and inflation stays at target. It is also the solution of conventional New Keynesian analysis, which linearizes the model around its shockless steady state. There, the fully stabilized allocation is the only bounded solution in a neighborhood of the steady state when the policy coefficients satisfy

$$\phi_y > -\rho, \qquad \phi_\pi + \frac{\rho}{(1+1/\eta)\kappa}\phi_y > 1. \tag{3.5}$$

The first condition is a trace condition on the linearized system, and the second one is the Taylor principle (Appendix C.1).[24] In the non-linear model, by contrast, these conditions do not rule out alternative bounded global solutions.

The second class provides such alternatives. It consists of stochastic solutions, indexed by shape constants $b, \zeta \in (0,1)$ satisfying the shape and end-point conditions (A.2) and (A.6) in Online Appendix A.2. At least one such solution exists for a combination of ϕ_π and ϕ_y satisfying (3.5).[25] Online Appendix A gives the construction and global proof. In each, centered inflation lies on an increasing graph of the output ratio q, and excess volatility σ_t^s is also a function of the output ratio, $\widehat{\pi}_t = \widehat{\pi}(q_t)$ and $\sigma_t^s = \sigma^s(q_t)$, where

$$\widehat{\pi}(q) = \underline{\widehat{\pi}} + \beta q^\zeta, \qquad \sigma^s(q) = \frac{\sqrt{\sigma^2 + 2(1+\zeta)F(q)} - \sigma}{1+\zeta}, \tag{3.6}$$

and F is the forcing function defined in equation (A.4) of Online Appendix A.2. Here $\underline{\widehat{\pi}}$ is the negative root of the drift of the Phillips curve (3.3b) at zero output and $\beta = -b\underline{\widehat{\pi}}$, so the graph rises from $\underline{\widehat{\pi}}$ to $(1-b)\underline{\widehat{\pi}} < 0$ at natural output. Excess volatility is positive on $(0, \bar{q})$ and vanishes at $\bar{q} < 1$, the smallest zero of F.

Lemma 2 (Stochastic solutions)**.** *For ϕ_π and ϕ_y satisfying (3.5), at least one pair (b, ζ) satisfies the shape and end-point conditions (A.2) and (A.6) of Online Appendix A.2. Let (b, ζ) be any pair satisfying these conditions, and let $q_0 \in (0, \bar{q})$. With $\widehat{\pi}_t = \widehat{\pi}(q_t)$ and $\sigma_t^s = \sigma^s(q_t)$, the IS equation (3.3a) has a unique strong solution q_t from q_0. Together with $\pi_t = \bar{\pi} + \widehat{\pi}(q_t)$ and $\sigma_t^\pi = \beta\zeta q_t^\zeta \sigma^s(q_t)$, it solves the New Keynesian Phillips curve (3.3b) and has the following properties:*

(i) Underproduction: *The output ratio remains in $(0, \bar{q})$ at all dates and states, so output stays*

[24] This local uniqueness rests on discarding the explosive solutions (Blanchard and Kahn, 1980). Cochrane (2011) argues that nothing in the economics justifies doing so: the discarded paths feature explosive inflation or deflation, and household transversality conditions restrict real wealth but do not rule out nominal explosions. Absent that restriction, the Taylor principle leaves inflation indeterminate.

[25] For each b, the shape conditions hold for all sufficiently small ζ, whatever the policy coefficients. The end-point conditions restrict the policy coefficients jointly with (b, ζ). Whenever the rule satisfies the Taylor principle (3.5), with or without the trace condition, they hold for all sufficiently small ζ, for every b if $\phi_y \geq 0$ and for b close to one otherwise (Lemma A.2). They also hold for some rules that violate the Taylor principle.

below its flexible-price level and the output gap stays below $\bar{q} - 1 < 0$.

(ii) Below-target inflation: *Inflation remains in the bounded band* $(\bar{\pi} + \underline{\widehat{\pi}}, \bar{\pi} + \widehat{\pi}(\bar{q}))$ *below target and moves together with the output gap.*

(iii) Excess volatility: $\sigma_t^s > 0$ *and* $\sigma_t^\pi > 0$ *almost surely, so the output ratio and inflation keep fluctuating, and output, with loading* $\sigma + \sigma_t^s$*, is more volatile than natural output.*

(iv) Stationarity: *The output ratio has a unique invariant distribution, with a positive density on* $(0, \bar{q})$*, and its law converges to this distribution in total variation from every* q_0*. The output gap, inflation and excess volatility therefore have nondegenerate stationary distributions, and distinct pairs* (b, ζ) *give distinct joint stationary laws.*

Proof. See Online Appendix A.3. □

Online Appendix A.7 records further properties of these stationary solutions, including bounds and cyclical comovement, stationary means and drift restrictions, convergence in distribution without pathwise stabilization and the role of the Frisch elasticity. Online Appendix A.8 shows that the family survives an occasionally binding effective lower bound on the policy rate.

Lemma 2 extends the multiplicity result in Lee and Dordal i Carreras (2026) to an exact New Keynesian Phillips curve. The underlying mechanism is the same: precautionary saving, together with the exact IS equation, supports a stationary family of equilibria with underproduction, below-target inflation and self-fulfilling excess volatility. The key difference is in the supply block. While Lee and Dordal i Carreras (2026) use a log-linearized Phillips curve for their analytical results and study the fully nonlinear pricing model numerically, we derive the Phillips curve directly from price setting à la Rotemberg (1982). This allows us to establish the multiplicity result analytically without linearizing either the marginal-cost term or the quadratic inflation term.

Figure 3 illustrates how the stochastic equilibrium (3.6) depends on the shape constants (b, ζ). All solutions start from the same centered inflation $\underline{\widehat{\pi}}$ at zero output, and b mainly governs how close inflation gets to target near $\bar{q}$, with little effect on excess volatility. A larger ζ steepens the graph at low output and lowers excess volatility throughout.

We now show that a passive money supply selects neither class of solutions: it accommodates the money demand of both the fully stabilized allocation and every stochastic solution in Lemma 2, leaving the limiting equilibrium globally indeterminate.

Proposition 1 (Passive money supply)**.** *Let the money supply be passive, satisfying* (3.4)*. The fully stabilized allocation of Lemma 1 is a limiting equilibrium in the sense of Definition 2, and so is every stochastic solution of Lemma 2. The limiting equilibrium is thus indeterminate whenever a pair* (b, ζ) *satisfies the shape and end-point conditions* (A.2) *and* (A.6) *in Online Appendix A.2. Such pairs of* (b, ζ) *exist whenever the policy rule* (2.20) *satisfies the Taylor principle. In particular,*

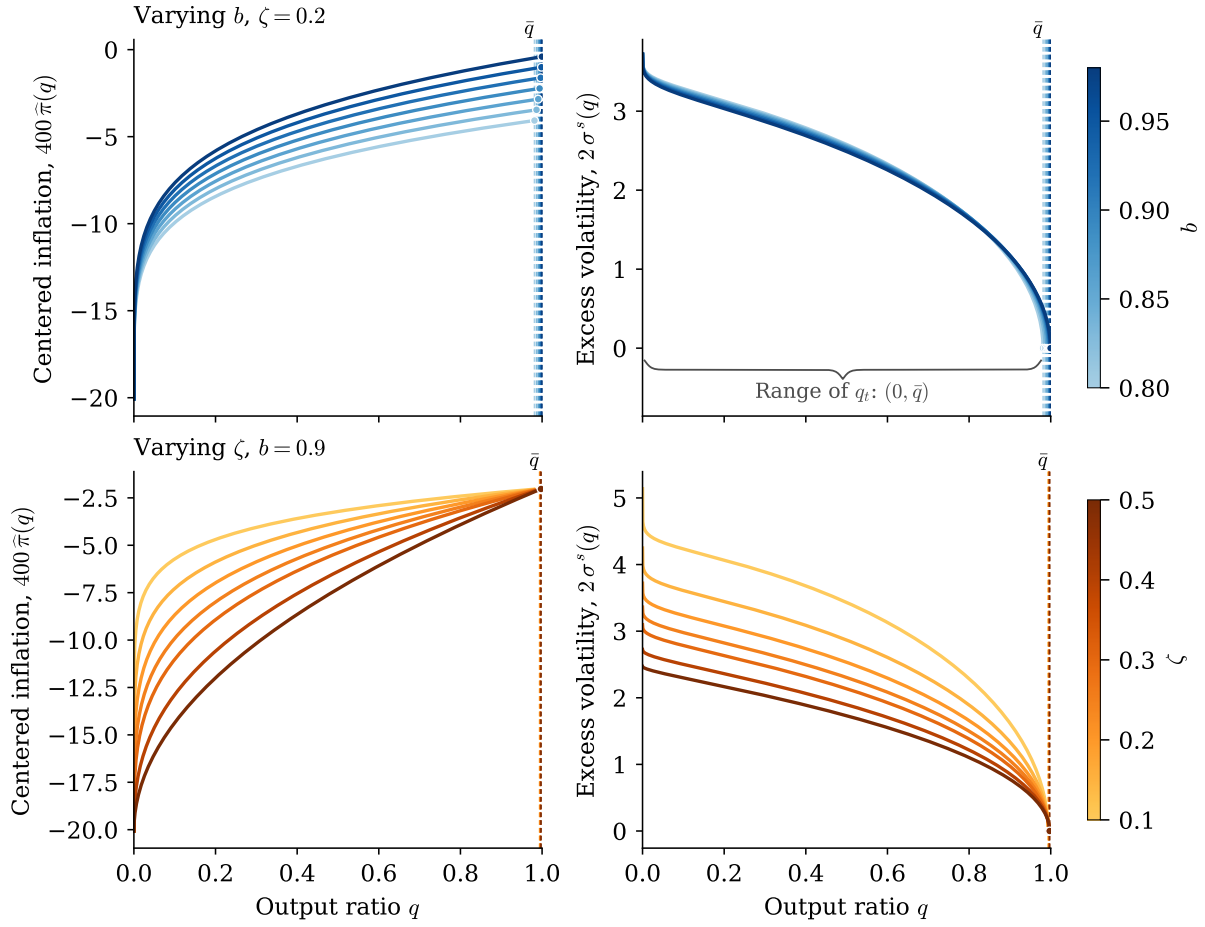


Figure 3: Centered inflation $\widehat{\pi}(q)$ and excess volatility $\sigma^s(q)$ in (3.6) for different shape constants. The top row varies b with $\zeta = 0.2$; the bottom row varies ζ with $b = 0.9$. Dashed vertical lines mark the upper end point $\bar{q}$ of each solution, which bounds the range $(0, \bar{q})$ of the output ratio. Quarterly calibration with $\rho = 0.01$, $\varepsilon = 7$, $\tau = 744.9$, $\sigma = 0.0022$, $\eta = 1$, $\phi_\pi = 1.5$ and $\phi_y = 0.5$; inflation is annualized in percentage points and volatility is per square-root year. Every pair satisfies the shape and end-point conditions (A.2) and (A.6).

every policy rule that makes the fully stabilized allocation locally determinate leaves the limiting equilibrium globally indeterminate.

Proof. Under passive supply (3.4), money-market clearing (3.3c) holds for any allocation, and initial balances adjust to $\tilde{m}_0 = \tilde{d}_0 Y_0$. Lemmas 1 and 2 provide the two classes of solutions to the IS equation (3.3a) and the Phillips curve (3.3b). Lemma A.2 in Online Appendix A.2 supplies shape pairs under the Taylor principle, which local determinacy requires (Appendix C.1). Online Appendix C.1 completes the argument. □

Under passive supply, the fully stabilized allocation therefore coexists with a two-parameter family of stochastic equilibria, each of which can start anywhere in $(0, \bar{q})$ and keeps output below

its natural level and inflation below target. Near the top of the interval, and throughout it when $\phi_y \geq 0$ and $\phi_\pi \geq 1$, the policy rule holds the real rate below the natural rate, $i_t - \pi_t < r^n$, which by itself would raise current demand. Excess volatility offsets this stimulus: it raises consumption risk, and the resulting precautionary term $\sigma_t^s(\sigma + \sigma_t^s)$ in the IS equation keeps current output low relative to its expected path.[26] Low output in turn lowers marginal cost, and the Phillips curve keeps inflation below target. Price stickiness is essential to this mechanism. With flexible prices and the same productivity risk, static optimal pricing keeps output at its natural level, and in the cashless limit the policy rule pins inflation to target for any $\phi_\pi \neq 1$ (Appendix C.2).

Recall that this indeterminacy is invisible to local analysis around the steady state. Under (3.5), the linearized model has two unstable roots and no predetermined variables. Any solution other than the fully stabilized allocation therefore diverges, leading the conventional criterion to classify that allocation as locally determinate. The same conclusion holds in the flexible-price, perfect-foresight economy of Woodford (1998), where an interest-rate rule delivers a locally unique equilibrium.[27] The stochastic solutions we construct, however, lie entirely below $\bar{q} < 1$ and never approach the fully stabilized allocation. They therefore remain outside the neighborhood in which the local analysis applies.

Note that the cashless economy itself, with $\gamma = 1$, is indeterminate in the same way. With money demand and supply both zero, money-market clearing (2.21) imposes no restriction, as under passive supply, and equilibrium again rests on the IS equation and the Phillips curve alone.

Corollary 1 (Cashless economy)**.** *When $\gamma = 1$, real money demand is zero. With zero money, the equilibria of the cashless economy coincide with the limiting equilibria under passive money supply.*

Proof. At $\gamma = 1$, $d_{1,t} = 0$, $H_{1,t} = 1$ and $P_{1,t} = p_t$ for any spread, so the IS equation and the Phillips curve coincide with their limits (3.3a) and (3.3b), and the natural rate equals r^n. Money demand $d_{1,t}Y_t$ vanishes. Thus, market clearing requires zero money and, as under passive supply, restricts no allocation. □

3.2 Full stabilization

Consider instead the rule that supplies limiting real money demand at natural output,

$$\tilde{m}_t^s = \tilde{d}_t Y_t^n. \tag{3.7}$$

[26] When $\phi_y < 0$, the real rate $i_t - \pi_t$ can exceed the natural rate near zero output, even under (3.5). There, the precautionary term reinforces the contractionary real rate. For this, see Lemma A.4 in Online Appendix A.7.

[27] Woodford (1998) pairs the interest-rate rule with a fiscal policy that keeps total nominal government liabilities constant, so the conventional deficit is zero in every period. Under other fiscal rules, the government's intertemporal budget constraint can determine the price level even under an interest-rate peg (Cochrane, 2023). In our economy, money is issued against household loans and household net wealth is zero, so no such fiscal condition arises.

With (3.7), money then no longer accommodates realized output but instead selects the fully stabilized solution. By the quantity equation (2.22), the rule can also be read as a commitment to supply the money that keeps real purchases at their natural level.

Proposition 2 (Full stabilization)**.** *Let $\tilde{m}_0 = \tilde{d}_0 Y_0^n$, let the money supply follow (3.7), and let ϕ_y be arbitrary.*

(i) Non-unit inflation response: *If $\phi_\pi \neq 1$, the unique limiting equilibrium is the fully stabilized one, $q_t = 1$, $\pi_t = \bar{\pi}$ and $i_t = r^n + \bar{\pi}$, with $\sigma_t^s = \sigma_t^\pi = 0$.*

(ii) Unit inflation response: *If $\phi_\pi = 1$, every limiting equilibrium still has $q_t = 1$, $\sigma_t^s = 0$ and $i_t = r^n + \pi_t$, but inflation is indeterminate: in deterministic equilibria, it either stays at target or starts above target and converges monotonically to $\bar{\pi} + 2\rho/(\varepsilon - 1)$, while stochastic equilibria include ones in which it has a nondegenerate stationary distribution above target.*

Proof. Clearing equation (3.3c) under (3.7) gives $q_t = 1$ at all dates, and the initial money stock is compatible by assumption. The loading and drift in (3.3a) give $\sigma_t^s = 0$ and $(\phi_\pi - 1)\widehat{\pi}_t = 0$ for any ϕ_y.

For part (i), let $\phi_\pi \neq 1$. Then $\widehat{\pi}_t = 0$ and $\sigma_t^\pi = 0$, so both sides of (3.3b) vanish and the rule (2.20) gives $i_t = r^n + \bar{\pi}$. Conversely, these processes satisfy (2.20) and (3.3), so they form the unique limiting equilibrium.

For part (ii), let $\phi_\pi = 1$. The drift condition $(\phi_\pi - 1)\widehat{\pi}_t = 0$ then holds for any inflation path, and the rule gives $i_t = r^n + \pi_t$. The limiting equilibria are therefore exactly the inflation paths that satisfy (3.3b) at $q_t = 1$. Appendix D characterizes all deterministic equilibria and constructs stochastic equilibria in which inflation has a nondegenerate stationary distribution. □

Full stabilization therefore does not rest on the Taylor principle once the issuer commits to the money supply (3.7). The money rule fixes the output ratio, and the IS equation then requires the real-rate deviation $(\phi_\pi - 1)\widehat{\pi}_t$ to vanish. This pins inflation to target for any $\phi_\pi \neq 1$, including interest-rate rules with $\phi_\pi < 1$, while the output coefficient plays no role. Only at $\phi_\pi = 1$, when the interest-rate rule sets the real rate at r^n regardless of inflation, does the IS equation leave inflation free. The Phillips curve alone then admits a continuum of inflation paths.[28]

3.3 Money supply and the stochastic equilibrium

Fix (b, ζ) satisfying the shape and end-point conditions (A.2) and (A.6) in Online Appendix A.2. By Lemma 2, such a pair exists whenever the policy coefficients satisfy (3.5). The graph (3.6) then maps $(0, \bar{q})$ onto the operating range $(\underline{\widehat{\pi}}, \widehat{\pi}(\bar{q}))$ of centered inflation. For $\pi_t - \bar{\pi}$ in this range,

[28] The flexible-price benchmark in Appendix C.2 imposes the same zero-gap restriction through static optimal pricing rather than a money supply commitment.

suppose that, for reasons outside the model, the money supply follows the rule

$$\tilde{m}_t^{\mathrm{s}} = \tilde{d}_t Y_t^n \left(\frac{\pi_t - \bar{\pi} - \underline{\hat{\pi}}}{\beta} \right)^{1/\zeta}, \qquad \sigma_t^{m/d} \geq 0, \tag{3.8}$$

where $\sigma_t^{m/d}$ is the loading of $d\log(\tilde{m}_t^{\mathrm{s}}/\tilde{d}_t)$ on dZ_t. Under the first part, the money supply $\tilde{m}_t^{\mathrm{s}}$ rises with inflation: it equals the money that households demand at the output ratio that the graph (3.6) associates with current inflation, $((\pi_t - \bar{\pi} - \underline{\hat{\pi}})/\beta)^{1/\zeta}$. Money-market clearing (3.3c) then holds only if output takes this value, so output and inflation stay on the graph, just as output stays at its natural level under the stabilizing rule (3.7). Since $\tilde{m}_t/\tilde{d}_t$ equals real purchases of goods by the quantity equation (2.22), the second part can be read as requiring the money supply to keep real purchases from falling in response to a positive productivity innovation. Together, the two parts select a single stochastic solution as the unique limiting equilibrium.

Proposition 3 (Stochastic equilibrium)**.** *Let (b, ζ) satisfy the shape and end-point conditions (A.2) and (A.6) in Online Appendix A.2, let $\tilde{m}_0 = \tilde{d}_0 Y_0^n q_0$ with $q_0 \in (0, \bar{q})$, and let the money supply follow (3.8). The unique limiting equilibrium is then the stochastic solution of Lemma 2 from q_0, with i_t given by (2.20).*

Proof. See Online Appendix C.2. □

Propositions 2 and 3 share a common structure. The stabilizing and stochastic rules both supply limiting money demand at natural output scaled by a factor, equal to one under (3.7) and to the inverse graph evaluated at current inflation under (3.8). In both, the inherited money stock fixes the initial output ratio, and each rule also pins down how the ratio loads on productivity innovations: under (3.7), which holds the ratio at one, and (3.8), through its diffusion loading restriction, respectively. Selection rests on both, because once the initial ratio and its loading are fixed, the IS equation (3.3a) determines the rest of its path.

Passive supply (3.4) instead sets the factor equal to the realized output ratio, so money-market clearing restricts no allocation and selects no equilibrium (Proposition 1). At $\gamma = 1$, no money is held at all, so the money supply cannot select an equilibrium either (Corollary 1). Selection through the money supply is therefore a property of the cashless limit, not of the cashless economy itself.

4 Conclusion

We show that the cashless limit of the New Keynesian model does not eliminate global equilibrium multiplicity. As the share of transactions requiring cash vanishes, real balances and the liquidity wedge disappear too, and both the IS equation and the Phillips curve converge to those of the exact cashless model. Yet under an interest-rate rule satisfying the Taylor principle, the fully stabilized

allocation can coexist globally with a two-parameter family of stationary stochastic equilibria. In these equilibria, output remains below its natural level and inflation below target. Self-fulfilling excess volatility sustains precautionary saving, keeping aggregate demand persistently low. We establish this multiplicity analytically using an exact nonlinear supply block, in contrast to our earlier work (Lee and Dordal i Carreras, 2026). The stochastic equilibria persist even when the effective lower bound on the policy rate occasionally binds.

The existence of these global equilibria raises the question of whether monetary policy can select among them as the economy becomes cashless. We show that it can, as long as the cash share remains positive. Although real money balances vanish in the limit, money demand normalized by the cash share remains positive, allowing money-market clearing to restrict equilibrium outcomes even when cash is used in an arbitrarily small fraction of transactions. A commitment to supply this normalized demand at natural output uniquely selects the fully stabilized allocation whenever the coefficient on inflation differs from one, regardless of whether the interest-rate rule satisfies the Taylor principle. Alternatively, an inflation-indexed money-supply rule, with a restriction that prevents real purchases from falling following a positive productivity innovation, selects a particular member of the stochastic equilibria. Both selection results require an initial money stock compatible with the selected initial output ratio. By contrast, a passive supply rule that accommodates realized money demand selects no equilibrium.

This selection role disappears at the exact cashless end-point. When the cash share is zero, money demand and money supply both vanish, and money-market clearing imposes no restriction on equilibrium allocations. The global multiplicity under the interest-rate rule therefore remains unresolved. The distinction is between an economy in which money can be used in an arbitrarily small fraction of transactions and one in which money is not used at all. While the equilibrium conditions converge to those of the cashless model, the equilibrium-selection mechanism does not. Money can become negligible as a means of payment and yet retain a decisive role in determining which equilibrium is realized.

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A Household optimality conditions

This section derives the household results of Sections 2.2 and 2.3: the demand structure (2.6) with price index (2.7), money demand (2.8), real expenditure, the household budget (2.11), the

labor condition (2.12), the discount factor and Euler condition (A.3), and the IS equation (2.15b) at natural output. Online Appendix B.1 states the admissible plans and a sufficient condition for global optimality.

A.1 First-order conditions and the allocation across locations

The dynamic budget (2.11) charges real expenditure, which presumes the allocation of Section 2.2. For arbitrary purchases and balances, the budget charges resource consumption and the carrying cost of money separately,

$$da_t^h = \left[r_t a_t^h - \delta_t m_t^h + \frac{w_t}{p_t} L_t^h + \frac{D_t}{p_t} + \mathcal{T}_t - C_t^h \right] dt, \tag{A.1}$$

because money earns i_t^m rather than i_t and every purchase lowers wealth by its price. Admissible deviations (Online Appendix B.1) obey this general budget constraint, and Section A.2 shows that it reduces to (2.11) at the allocation of Section 2.2.

Let $\mu_t^h > 0$ be the marginal value of real wealth, the multiplier on (A.1), and $\psi_t^h \geq 0$ the multiplier on the cash constraint (2.4). By (2.2), the marginal utility of consumption at location k is $\partial \log \mathcal{C}_t^h / \partial c_t^h(k) = (c_t^h(k)/\mathcal{C}_t^h)^{-1/\theta}/\mathcal{C}_t^h$. A unit of consumption at any location lowers wealth by one unit, valued at μ_t^h, and a unit at a cash location also uses one unit of the cash constraint, valued at ψ_t^h. One more unit of real balances costs δ_t in the budget and relaxes the cash constraint by v. The marginal disutility of labor is $(L_t^h)^{1/\eta}$ and the value of its real wage is $\mu_t^h w_t/p_t$. The first-order conditions for consumption at each location, labor and real balances are therefore

$$\begin{aligned} \frac{1}{\mathcal{C}_t^h} \left(\frac{c_t^h(k)}{\mathcal{C}_t^h} \right)^{-1/\theta} &= \begin{cases} \mu_t^h + \psi_t^h, & k \in \mathcal{K}_{\text{cash}}^h, \\ \mu_t^h, & k \in \mathcal{K}_{\text{credit}}^h, \end{cases} \\ (L_t^h)^{1/\eta} &= \mu_t^h \frac{w_t}{p_t}, \\ v\psi_t^h &= \delta_t \mu_t^h. \end{aligned} \tag{A.2}$$

The left side of (A.2) is strictly decreasing in $c_t^h(k)$ and unbounded as $c_t^h(k)$ approaches zero, so the household buys a positive quantity at every location. The right side takes one value at all cash locations and another at all credit locations. The household therefore buys the same quantity $X_t^h/(1-\gamma)$ at every cash location and the same quantity V_t^h/γ at every credit location, as stated in Section 2.2.

Dividing the balances condition in (A.2) by $v\mu_t^h$ gives the ratio of the multipliers, and with it the ratio of the right sides of the two consumption conditions,

$$\frac{\psi_t^h}{\mu_t^h} = \frac{\delta_t}{v}, \qquad \frac{\mu_t^h + \psi_t^h}{\mu_t^h} = 1 + \frac{\delta_t}{v} = R_t,$$

where the last equality is (2.5) of the relative cost. Dividing the cash-location condition by the credit-location condition therefore gives

$$\left(\frac{c_t^h(k)}{c_t^h(k')}\right)^{-1/\theta} = R_t, \qquad k \in \mathcal{K}_{\text{cash}}^h, \quad k' \in \mathcal{K}_{\text{credit}}^h,$$

so the quantity at a cash location is $R_t^{-\theta}$ times the quantity at a credit location. Substituting the common quantities $X_t^h/(1-\gamma)$ and V_t^h/γ into the basket (2.2) gives the reduced index

$$\mathcal{C}_t^h = \left[(1-\gamma)^{1/\theta}(X_t^h)^{1-1/\theta} + \gamma^{1/\theta}(V_t^h)^{1-1/\theta}\right]^{\theta/(\theta-1)}.$$

Write $\mathcal{C}(X, V)$ for this index as a function of cash and credit purchases. Because $\log \mathcal{C}_t^h$ is concave in purchases and the constraints are linear, the consumption conditions in (A.2) are sufficient and necessary, given resource consumption and money balances. Hence $\mathcal{C}(X, V)$ is the largest basket attainable with cash purchases X and credit purchases V.

A.2 Money demand and expenditure

With a positive spread $\delta_t > 0$ the cash constraint binds, and with a zero spread it binds by the tie-breaking assumption in Section 2.2, so $m_t^h = X_t^h/v$. The quantity ratio of Section A.1 and the even split give $X_t^h/(1-\gamma) = R_t^{-\theta}V_t^h/\gamma$. Replacing V_t^h by $C_t^h - X_t^h$ and collecting terms,

$$X_t^h\left[\gamma + (1-\gamma)R_t^{-\theta}\right] = (1-\gamma)R_t^{-\theta}C_t^h,$$

and therefore

$$X_t^h = \frac{(1-\gamma)C_t^h}{1-\gamma+\gamma R_t^{\theta}} = vd_{\gamma,t}C_t^h,$$

with $d_{\gamma,t}$ given in (2.8). The binding cash constraint (2.4) and $V_t^h = C_t^h - X_t^h$ give real money demand $m_t^{h,\mathrm{d}} = m_t^h = d_{\gamma,t}C_t^h$ and $V_t^h = [1 - vd_{\gamma,t}]C_t^h$, the remaining equalities in (2.8). For finite positive spreads both payment classes have strictly positive purchases, and $vd_{\gamma,t} < 1-\gamma$.

Solving the consumption conditions in (A.2), with right sides $R_t\mu_t^h$ and μ_t^h, gives

$$c_t^h(k) = (R_t\mu_t^h\mathcal{C}_t^h)^{-\theta}\mathcal{C}_t^h, \quad k \in \mathcal{K}_{\text{cash}}^h, \qquad c_t^h(k) = (\mu_t^h\mathcal{C}_t^h)^{-\theta}\mathcal{C}_t^h, \quad k \in \mathcal{K}_{\text{credit}}^h.$$

Substituting these quantities into the basket (2.2), where cash and credit locations have measures $1-\gamma$ and γ, and solving for μ_t^h gives

$$\mu_t^h = \frac{1}{\mathcal{C}_t^h}\left[\gamma + (1-\gamma)R_t^{1-\theta}\right]^{-1/(1-\theta)} = \frac{p_t}{P_{\gamma,t}\mathcal{C}_t^h} = \frac{1}{Q_t^h},$$

where the second equality is the definition (2.7) of the price index and the third is the definition of

real expenditure in Section 2.2. Note that this is the marginal value of real wealth in Section 2.3.

Discounted at rate ρ, it gives the stochastic discount factor $\xi_t^h = e^{-\rho t}/Q_t^h$ used in the Euler condition (2.13) and, in symmetric equilibrium, in the admissible-domain conditions (B.1) of Online Appendix B.1. Substituting this value of μ_t^h into the location demands gives

$$c_t^h(k) = \left(\frac{p_t R_t}{P_{\gamma,t}}\right)^{-\theta} \mathcal{C}_t^h, \quad k \in \mathcal{K}_{\text{cash}}^h, \qquad c_t^h(k) = \left(\frac{p_t}{P_{\gamma,t}}\right)^{-\theta} \mathcal{C}_t^h, \quad k \in \mathcal{K}_{\text{credit}}^h,$$

which are the demand structure (2.6), and substituting them into the labor condition (A.2) gives

$$\frac{1}{Q_t^h}\frac{w_t}{p_t} = (L_t^h)^{1/\eta},$$

which is the intratemporal condition (2.12).

Real expenditure Q_t^h equals resource consumption plus the carrying cost of money. To see this, multiply each location's consumption condition in (A.2) by $c_t^h(k)$ and integrate over locations. On the right, cash locations contribute $R_t \mu_t^h X_t^h$ and credit locations contribute $\mu_t^h V_t^h$. On the left,

$$\begin{aligned}\int_0^1 \frac{1}{\mathcal{C}_t^h}\left(\frac{c_t^h(k)}{\mathcal{C}_t^h}\right)^{-1/\theta} c_t^h(k)\,dk &= (\mathcal{C}_t^h)^{1/\theta-1}\int_0^1 c_t^h(k)^{1-1/\theta}dk \\ &= (\mathcal{C}_t^h)^{1/\theta-1}(\mathcal{C}_t^h)^{(\theta-1)/\theta} = 1\end{aligned}$$

from (2.2). Hence $R_t X_t^h + V_t^h = 1/\mu_t^h = Q_t^h$. Using $R_t = 1 + \delta_t/v$, $m_t^h = X_t^h/v$ and the money demand (2.8),

$$R_t X_t^h + V_t^h = X_t^h + V_t^h + \delta_t m_t^h = C_t^h + \delta_t m_t^h = C_t^h(1 + \delta_t d_{\gamma,t}) = C_t^h H_{\gamma,t}.$$

Real expenditure therefore takes the three forms displayed in Section 2.2,

$$Q_t^h = \frac{P_{\gamma,t}\mathcal{C}_t^h}{p_t} = C_t^h + \delta_t m_t^h = C_t^h H_{\gamma,t}.$$

Because $C_t^h + \delta_t m_t^h = Q_t^h$, the terms $-\delta_t m_t^h - C_t^h$ in the budget constraint (A.1) add to $-Q_t^h$, so (A.1) becomes (2.11).

No other plan attains the same basket at lower cost. For any plan, $m_t^h \geq X_t^h/v$ and $\delta_t \geq 0$ give $C_t^h + \delta_t m_t^h \geq R_t X_t^h + V_t^h$. Because the index (2.2) is concave and homogeneous of degree one, the CES demands (2.6) minimize $R_t X_t^h + V_t^h$ among the allocations that attain $\mathcal{C}_t^h$, with minimum $P_{\gamma,t}\mathcal{C}_t^h/p_t$. Hence $C_t^h + \delta_t m_t^h \geq P_{\gamma,t}\mathcal{C}_t^h/p_t$, with equality at the allocation of Section 2.2. Any other plan for the same basket and labor therefore leaves weakly less wealth.

A.3 Natural output and the IS equation

Let $\xi_t^h = e^{-\rho t}/Q_t^h$ be the household's stochastic discount factor, the marginal value of real wealth from Appendix A.2 discounted at rate ρ, and let $\xi_t^{N,h} = \xi_t^h/p_t$ be its nominal counterpart. At an interior optimum the short-account Euler condition is

$$\mathbb{E}_t[d\xi_t^{N,h}] = -i_t \xi_t^{N,h} dt. \tag{A.3}$$

Let $\mathrm{mc}_t = w_t/(p_t A_t)$ be real marginal cost. In symmetry, $Q_t = Y_t H_{\gamma,t}$ and $L_t = Y_t/A_t$, so the labor condition (2.12) gives

$$\mathrm{mc}_t = H_{\gamma,t} L_t^{1+1/\eta}. \tag{A.4}$$

Flexible-price producers set $\mathrm{mc}_t^n = \alpha$; solving for labor at the actual wedge yields L_t^n in (2.14). Since $L_t = L_t^n q_t$, it follows that $\mathrm{mc}_t/\alpha = q_t^{1+1/\eta}$. Natural real expenditure, real expenditure at natural output, is therefore $Q_t^n = Y_t^n H_{\gamma,t} = A_t \alpha^{\eta/(1+\eta)} H_{\gamma,t}^{1/(1+\eta)}$. Taking logs and using (2.3) and (2.10),

$$d\log Q_t^n = \left[g - \frac{\sigma^2}{2} + \frac{g_{H,t}}{1+\eta} \right] dt + \sigma_t^n dZ_t. \tag{A.5}$$

For this calculation, write $dq_t/q_t = b_t^q dt + \sigma_t^s dZ_t$ and $dQ_t/Q_t = b_t^Q dt + \sigma_t^Q dZ_t$, where b_t^q and b_t^Q denote relative drifts. The product $Q_t = q_t Q_t^n$ gives

$$b_t^Q = b_t^q + g - \frac{\sigma^2}{2} + \frac{g_{H,t}}{1+\eta} + \frac{(\sigma_t^n)^2}{2} + \sigma_t^n \sigma_t^s, \qquad \sigma_t^Q = \sigma_t^n + \sigma_t^s. \tag{A.6}$$

Itô's formula for the nominal discount factor gives

$$\frac{d\xi_t^N}{\xi_t^N} = \left[-\rho - \pi_t - b_t^Q + (\sigma_t^Q)^2 \right] dt - \sigma_t^Q dZ_t. \tag{A.7}$$

The Euler condition requires $b_t^Q = i_t - \pi_t - \rho + (\sigma_t^Q)^2$. Substituting (A.6) and collecting terms gives the natural rate (2.15a) and the IS equation (2.15b). This calculation retains the covariance between the output ratio and natural real expenditure, as well as the variance term from inverse expenditure.

B Producer optimality conditions

This section sets up the producer's pricing problem, derives its individual first-order conditions and then imposes symmetry to obtain the Phillips curve (2.18). Online Appendix B.2 states the admissible pricing plans and a sufficient condition for global individual optimality, and Online

Appendix B.3 verifies it for the limiting equilibria.

B.1 Pricing plans and the normalized objective

Producer j chooses any progressively measurable, locally square-integrable inflation path $\{\pi_s^j\}_{s\geq t}$, so its relative price stays positive, finite and absolutely continuous.

The problem is easiest to study in terms of relative revenue $\Omega_t^j = (p_t^j/p_t)^{1-\varepsilon}$, the producer's real revenue as a fraction of aggregate consumption, and the control $u_t^j = (\pi_t^j - \bar{\pi})\Omega_t^j$, inflation in excess of indexation weighted by relative revenue. By the demand curve in (2.1), real revenue is $\Omega_t^j C_t$, output is $Y_t^j = (\Omega_t^j)^{1/\alpha} C_t$, and the real fee Θ_t^j/p_t is $\tau(u_t^j)^2 C_t/(2\Omega_t^j)$. Relative revenue evolves as

$$\dot{\Omega}_t^j = (\varepsilon - 1)(\widehat{\pi}_t \Omega_t^j - u_t^j), \tag{B.1}$$

so a producer whose price rises faster than the price level loses revenue share. No symmetry has been imposed.

Labor is $L_t^j = Y_t^j/A_t$, so the real wage bill is $\mathrm{mc}_t \cdot C_t(\Omega_t^j)^{1/\alpha}$, with real marginal cost (A.4). Real profit is therefore $Q_t f_t(\Omega_t^j, u_t^j)$, and since $\xi_s/\xi_t = e^{-\rho(s-t)}\mu_s/\mu_t$ with $\mu_t = 1/Q_t$ (Section 2.3 and Appendix A.2), the objective is a positive multiple of an objective in utility units discounted at rate ρ:

$$\mathbb{E}_t \int_t^\infty \frac{\xi_s}{\xi_t} \frac{\Psi_s^j}{p_s} ds = Q_t \, \mathbb{E}_t \int_t^\infty e^{-\rho(s-t)} f_s(\Omega_s^j, u_s^j) ds, \tag{B.2}$$

where

$$f_t(\Omega, u) = \frac{\Omega}{H_{\gamma,t}} - L_t^{1+1/\eta} \Omega^{1/\alpha} - \frac{\tau}{2H_{\gamma,t}} \frac{u^2}{\Omega}.$$

B.2 First-order conditions and symmetry

Let λ_t^j be the current-value costate of relative revenue, its shadow value in the normalized problem. With the current-value Hamiltonian $f_t(\Omega, u) + \lambda(\varepsilon - 1)(\widehat{\pi}_t \Omega - u)$, producer j's first-order conditions are

$$\begin{aligned} \lambda_t^j &= -\frac{\tau u_t^j}{(\varepsilon - 1) H_{\gamma,t} \Omega_t^j}, \\ d\lambda_t^j &= [\rho \lambda_t^j - f_{t,\Omega}(\Omega_t^j, u_t^j) - (\varepsilon - 1)\widehat{\pi}_t \lambda_t^j] dt + dN_t^j, \end{aligned} \tag{B.3}$$

where N^j is a local martingale. The first condition balances the marginal fee from faster price growth against the value of the forgone relative revenue, and the second describes how that value evolves.

Only now impose symmetry, $\Omega_t^j = 1$ and $u_t^j = \widehat{\pi}_t$. The first condition becomes $\lambda_t = -\tau\widehat{\pi}_t/[(\varepsilon - 1)H_{\gamma,t}]$. At the symmetric point, $L_t = L_t^n q_t$ and (2.14) give $f_{t,\Omega}(1, \widehat{\pi}_t) = [1 - q_t^{1+1/\eta} + \tau\widehat{\pi}_t^2/2]/H_{\gamma,t}$ and $(\varepsilon - 1)\widehat{\pi}_t \lambda_t = -\tau\widehat{\pi}_t^2/H_{\gamma,t}$. In the Brownian equilibrium class of Section 2.3, let σ_t^λ denote

the loading of the symmetric costate on Z. Its relation to the inflation loading is

$$\sigma_t^\lambda = -\frac{\tau}{(\varepsilon-1)H_{\gamma,t}}(\sigma_t^\pi - \widehat{\pi}_t\sigma_t^H). \tag{B.4}$$

Equation (2.10) implies $dH_{\gamma,t}/H_{\gamma,t} = [g_{H,t}+(\sigma_t^H)^2/2]dt+\sigma_t^H dZ_t$. Apply the product rule directly to inflation relative to indexation:

$$d\widehat{\pi}_t = -\frac{\varepsilon-1}{\tau}\left[H_{\gamma,t}d\lambda_t + \lambda_t dH_{\gamma,t} + H_{\gamma,t}\sigma_t^H\sigma_t^\lambda dt\right]. \tag{B.5}$$

Substituting the symmetric costate drift in (B.3), (B.4) and the wedge dynamics gives

$$d\widehat{\pi}_t = \left[\left(\rho + g_{H,t} - \frac{(\sigma_t^H)^2}{2}\right)\widehat{\pi}_t - \kappa\left(q_t^{1+1/\eta}-1\right) - \frac{\varepsilon-1}{2}\widehat{\pi}_t^2 + \sigma_t^H\sigma_t^\pi\right]dt + \sigma_t^\pi dZ_t. \tag{B.6}$$

This is the Phillips curve (2.18). Note in (B.6) that the product's quadratic covariation produces the term involving $\sigma_t^H\sigma_t^\pi$ and, together with the conversion from $\log H_{\gamma,t}$, the variance correction in the coefficient on $\widehat{\pi}_t$. Both disappear when the wedge $H_{\gamma,t}$ is absolutely continuous ($\sigma_t^H = 0$).

C Local determinacy and the flexible-price benchmark

This section derives the local determinacy conditions (3.5) and the flexible-price benchmark of Section 3.1.

C.1 Local determinacy

When linearized around the fully stabilized equilibrium, with zero loadings, the drifts of (3.3a) and (3.3b) give

$$\frac{d}{dt}\begin{pmatrix} q_t - 1 \\ \widehat{\pi}_t \end{pmatrix} = \begin{pmatrix} \phi_y & \phi_\pi - 1 \\ -(1+1/\eta)\kappa & \rho \end{pmatrix}\begin{pmatrix} q_t - 1 \\ \widehat{\pi}_t \end{pmatrix}.$$

The coefficient matrix has trace $\phi_y + \rho$ and determinant $\rho\phi_y + (1+1/\eta)\kappa(\phi_\pi - 1)$. Both of its roots have positive real parts if and only if the trace and the determinant are positive, that is, if and only if the policy coefficients satisfy the trace condition and the Taylor principle in (3.5). The left side of the Taylor principle is the response of the nominal rate to a permanent increase in inflation along the long-run Phillips curve of the linearized model, $\rho\widehat{\pi} = (1+1/\eta)\kappa(q-1)$. Every solution other than the fully stabilized allocation then diverges (Blanchard and Kahn, 1980), and since neither the output ratio nor inflation is predetermined, the fully stabilized allocation is the only bounded solution of the linearized system. Otherwise, at least one root has a nonpositive real part, and the linearized system has a continuum of bounded solutions. The Taylor principle alone does not ensure local determinacy: if $\phi_y \leq -\rho$, the trace is nonpositive even when the

determinant is positive.

Note that Lemma A.2 in Online Appendix B.2 requires only the Taylor principle, so every locally determinate rule admits stochastic solutions for all sufficiently small ζ, with every b when $\phi_y \geq 0$ and with b close to one when $-\rho < \phi_y < 0$.

C.2 The stochastic flexible-price benchmark

Remove adjustment fees and inherited individual prices, retaining the stochastic fundamentals (2.3), payment institution and the financial domain (B.1) of Online Appendix B.1. All objects below refer to the flexible-price economy at its specified nonnegative spread.

Producer j chooses $z_t^j = p_t^j/p_t$ to maximize $C_t[(z_t^j)^{1-\varepsilon} - \mathrm{mc}_t(z_t^j)^{-\varepsilon}]$ pointwise. Its derivative changes from positive to negative at $z_t^j = \mathrm{mc}_t/\alpha$. Symmetry then gives $\mathrm{mc}_t = \alpha$, and the labor condition yields (2.14). Hence, $q_t = 1$ and (2.15b) gives $\sigma_t^s = 0$. The IS equation and Phillips curve give

$$i_t - \pi_t = r_t^n = \rho + g + \frac{g_{H,t}}{1+\eta} - \frac{\sigma^2 + (\sigma_t^n)^2}{2}, \tag{C.1a}$$

$$q_t = 1. \tag{C.1b}$$

Note that equation (C.1) fixes the real allocation and real rate at given productivity and spread, for arbitrary policy coefficients. Substitution into (2.20) gives

$$(\phi_\pi - 1)\widehat{\pi}_t = r_t^n - r^n. \tag{C.2}$$

The right side of (C.2) is zero when the liquidity wedge $H_{\gamma,t}$ is constant, as at a constant spread, and vanishes in the cashless limit. Whenever $\phi_\pi \neq 1$, inflation is $\pi_t = \bar{\pi} + (r_t^n - r^n)/(\phi_\pi - 1)$, regardless of ϕ_y, and it equals the target $\bar{\pi}$ when the wedge is constant. At the knife edge $\phi_\pi = 1$, a flexible-price equilibrium requires $r_t^n = r^n$ at all dates. Real quantities and the real rate then remain determined, but the Euler condition and policy rule leave inflation unrestricted. Since the output gap is identically zero, the Taylor principle reduces to $\phi_\pi > 1$, which excludes this case but is not necessary.

In the cashless limit of Section 3, the right side of (C.2) vanishes, so inflation is at target whenever $\phi_\pi \neq 1$, and

$$Y_t = C_t = A_t \alpha^{\eta/(1+\eta)}, \qquad L_t = \alpha^{\eta/(1+\eta)}, \qquad i_t - \pi_t = \rho + g - \sigma^2. \tag{C.3}$$

With $q_t = 1$, passive and stabilizing money supply coincide whenever the initial stock is compatible. The same real allocation and inflation conclusions hold in the cashless economy itself. Productivity risk remains in the level of output and in the natural rate, but not in the output ratio; when $\phi_\pi \neq 1$, it does not remain in inflation either.

D Inflation under the stabilizing rule at a unit inflation coefficient

We complete the proof of part (ii) of Proposition 2. At $q_t = 1$, the Phillips curve (3.3b) reads

$$d\widehat{\pi}_t = \widehat{\pi}_t \left(\rho - \frac{\varepsilon - 1}{2} \widehat{\pi}_t \right) dt + \sigma_t^{\pi} dZ_t. \tag{D.1}$$

Any progressively measurable loading σ_t^{π} for which (D.1) has a continuous solution defined at all dates yields a limiting equilibrium, with $q_t = 1$, $\sigma_t^s = 0$ and $i_t = r^n + \pi_t$. Let $\widehat{\pi}^* = 2\rho/(\varepsilon - 1)$, the positive zero of the drift, and let $\widehat{\pi}^\circ$ denote the initial value of $\widehat{\pi}_t$.

In a deterministic equilibrium, $\sigma_t^{\pi} = 0$ and (D.1) is the logistic equation. From $\widehat{\pi}^\circ = 0$ its solution is $\widehat{\pi}_t = 0$. From $\widehat{\pi}^\circ \neq 0$ it is

$$\widehat{\pi}_t = \frac{\widehat{\pi}^*}{1 + (\widehat{\pi}^*/\widehat{\pi}^\circ - 1) e^{-\rho t}}.$$

If $\widehat{\pi}^\circ > 0$, then $\widehat{\pi}^*/\widehat{\pi}^\circ - 1 > -1$, so the denominator stays positive and converges monotonically to one. Inflation therefore converges monotonically to $\widehat{\pi}^*$, and stays there if $\widehat{\pi}^\circ = \widehat{\pi}^*$. If $\widehat{\pi}^\circ < 0$, then $\widehat{\pi}^*/\widehat{\pi}^\circ - 1 < -1$, so the denominator vanishes at $t = \rho^{-1} \log(1 - \widehat{\pi}^*/\widehat{\pi}^\circ)$, where $\widehat{\pi}_t$ diverges to $-\infty$. No such path is an equilibrium.

Stochastic equilibria need not converge. Let $\sigma_t^{\pi} = \varsigma \widehat{\pi}_t$ with $0 < \varsigma^2 < 2\rho$, and let $\widehat{\pi}^\circ > 0$. Then (D.1) becomes the stochastic logistic equation

$$d\widehat{\pi}_t = \widehat{\pi}_t \left(\rho - \frac{\varepsilon - 1}{2} \widehat{\pi}_t \right) dt + \varsigma \widehat{\pi}_t \, dZ_t.$$

To solve it, introduce the geometric Brownian motion

$$M_t = \exp\left((\rho - \varsigma^2/2)t + \varsigma Z_t \right),$$

so that $dM_t = \rho M_t \, dt + \varsigma M_t \, dZ_t$ with $M_0 = 1$. Seek a solution of the form $\widehat{\pi}_t = M_t / V_t$. If V has paths of finite variation, Itô's product rule gives

$$d\widehat{\pi}_t = \rho \widehat{\pi}_t \, dt + \varsigma \widehat{\pi}_t \, dZ_t - \widehat{\pi}_t \frac{dV_t}{V_t}. \tag{D.2}$$

Matching the drift in (D.2) with (D.1) requires $dV_t/V_t = (\varepsilon - 1)\widehat{\pi}_t \, dt/2$, or $dV_t = (\varepsilon - 1) M_t \, dt/2$. With $\widehat{\pi}_0 = \widehat{\pi}^\circ$ one has $V_0 = 1/\widehat{\pi}^\circ$, and therefore

$$V_t = \frac{1}{\widehat{\pi}^\circ} + \frac{\varepsilon - 1}{2} \int_0^t M_s \, ds.$$

Substituting M yields the unique strong solution

$$\widehat{\pi}_t = \frac{\widehat{\pi}^\circ \exp\left((\rho - \varsigma^2/2)t + \varsigma Z_t\right)}{1 + \frac{\varepsilon-1}{2}\widehat{\pi}^\circ \int_0^t \exp\left((\rho - \varsigma^2/2)s + \varsigma Z_s\right) ds}.$$

The process is positive and finite at all dates. It is the classical stochastic logistic diffusion, whose unique stationary density is a gamma density on $(0, \infty)$, here proportional to $y^{2\rho/\varsigma^2 - 2} \exp(-(\varepsilon - 1)y/\varsigma^2)$ and therefore of shape $2\rho/\varsigma^2 - 1 > 0$ and rate $(\varepsilon - 1)/\varsigma^2$ (Capocelli and Ricciardi, 1974). Inflation is thus ergodic, with a nondegenerate stationary distribution above target, and does not converge.

No Doing Without Money: Global Indeterminacy and Equilibrium Selection in the New Keynesian Model

Marc Dordal i Carreras Seung Joo Lee

Main paper

A The stochastic equilibria

This online appendix constructs the stochastic solutions of Lemma 2 and proves their properties. Section A.1 presents notation. Sections A.2 and A.3 construct the family and prove Lemma 2, using the global properties of the state process stated in Section A.4 and proved in Section A.5. Section A.6 distinguishes stationarity from initialization. Section A.7 records further properties of the family, and Section A.8 shows that it survives an effective lower bound on the policy rate. Sections A.9–A.11 collect the diffusion calculations used in the proof that are not special to our model.

Online Appendix B states the admissibility domains and sufficient conditions for individual optimality and verifies them for the equilibria of the main text. Online Appendix C proves the money-supply results in Propositions 1 and 3. The equilibrium concept is the limiting equilibrium of Definition 2.

A.1 Notation and the limiting system

We work in the cashless limit of Section 3. Throughout, $\kappa = (\varepsilon - 1)/\tau$ and $\alpha = (\varepsilon - 1)/\varepsilon$. Natural labor and output are $L^n = \alpha^{\eta/(1+\eta)}$ and $Y_t^n = A_t L^n$, and the natural rate is $r^n = \rho + g - \sigma^2$, the intercept of the policy rule in (2.20). The state is the gross output ratio $q_t = Y_t / Y_t^n$, so the proportional output gap equals $q_t - 1$. Centered inflation $\widehat{\pi}_t = \pi_t - \bar{\pi}$ is inflation in excess of the common target and indexation rate. The policy rule sets $i_t = r^n + \bar{\pi} + \phi_y(q_t - 1) + \phi_\pi \widehat{\pi}_t$, the real rate is $r_t = i_t - \pi_t$, and the IS equation and the Phillips curve are equations (3.3a) and (3.3b). These are the exact equations of the main paper, without linearization. The construction below describes centered inflation, not actual inflation, and below-target inflation need not be deflation when $\bar{\pi} > 0$. The policy coefficients ϕ_y and ϕ_π are restricted only through the shape and endpoint conditions (A.2) and (A.6) below.

A.2 The construction domain

The construction indexes centered inflation by two shape constants, b and ζ, each between zero and one. They label members of the stochastic family, with household preferences and the policy rule in (2.20) held fixed. The level b scales the rise of centered inflation from its lower limit

toward zero, and the power ζ governs the curvature of that rise. The family uses four constants: the auxiliary rate ω, the lower limit $\underline{\widehat{\pi}}$ of centered inflation, the amplitude β and the limiting real-rate shortfall Φ,

$$\omega = \sqrt{\rho^2 + 2(\varepsilon - 1)\kappa}, \qquad \underline{\widehat{\pi}} = \frac{\rho - \omega}{\varepsilon - 1} < 0,$$
$$\beta = -b\underline{\widehat{\pi}} > 0, \qquad \Phi = \phi_y - (\phi_\pi - 1)\underline{\widehat{\pi}}. \tag{A.1}$$

We impose the following sufficient restrictions for this construction:

$$0 < b < 1, \qquad 0 < \zeta < 1, \qquad -\zeta\omega < \zeta^2\Phi < \omega,$$
$$\beta\zeta(\phi_\pi - 1) < \rho - \frac{\varepsilon - 1}{2}(1 - b)\underline{\widehat{\pi}}. \tag{A.2}$$

For fixed primitives and policy coefficients and for each fixed $b \in (0, 1)$, every sufficiently small positive ζ satisfies these inequalities (Lemma A.1). Thus the construction does not require changing preferences or the policy rule across members of the family. The role of each inequality appears below.

For a positive output ratio q, the centered-inflation graph is $\widehat{\pi}(q) = \underline{\widehat{\pi}} + \beta q^\zeta$, as in equation (3.6). Define the real-rate deviation $J(q) = i_t - \pi_t - r^n$ that the policy rule sets along the graph, and the forcing function $F(q)$, by

$$J(q) = \phi_y(q - 1) + (\phi_\pi - 1)\widehat{\pi}(q), \tag{A.3}$$
$$F(q) = \frac{\omega}{\zeta} - \frac{\kappa}{\beta\zeta}q^{1+1/\eta-\zeta} - \frac{(\varepsilon - 1)\beta}{2\zeta}q^\zeta - J(q). \tag{A.4}$$

The lower limit $\underline{\widehat{\pi}}$ is the negative root of the drift of the Phillips curve in (3.3b) at zero output,[1]

$$\rho\underline{\widehat{\pi}} + \kappa - \frac{\varepsilon - 1}{2}\underline{\widehat{\pi}}^2 = 0, \qquad \rho - (\varepsilon - 1)\underline{\widehat{\pi}} = \omega.$$

Lemma A.1 below shows that F has a smallest zero $\bar{q} \in (0, 1)$. The last inequality in (A.2) is exactly the condition $F(1) < 0$, so the state interval ends at $\bar{q}$, strictly below natural output. In other words, the state space is $q \in (0, \bar{q})$.[2] The inequality $-\zeta\omega < \zeta^2\Phi$ makes F positive near zero; it holds automatically when $\Phi \geq 0$, in particular when $\phi_y \geq 0$ and $\phi_\pi \geq 1$. The inequality $\zeta^2\Phi < \omega$ makes the drift of the log output ratio positive near zero, which is part of the boundary argument in Section A.5.

Lemma A.1 (Domain)**.** *Under (A.2), F has a smallest zero $\bar{q} \in (0, 1)$, and $F(q) > 0$, $\widehat{\pi}(q) < 0$ and $\sigma^s(q) > 0$ on $(0, \bar{q})$. For each $b \in (0, 1)$, (A.2) holds for all sufficiently small $\zeta > 0$. If $\phi_y \geq 0$ and $\phi_\pi \geq 1$, not both with equality, then F is strictly decreasing on $(0, 1)$, $\bar{q}$ is its unique zero*

[1] Choosing the negative root selects the family with below-target inflation.

[2] Sections A.4 and A.5 show that neither 0 nor $\bar{q}$ is reached in finite time from an interior point.

there, and J is strictly negative on $(0,1)$.

Proof. The powers in (A.4) vanish at zero, because $\zeta > 0$ and $1 + 1/\eta - \zeta > 0$, and $J(0+) = -\Phi$. Hence $F(0+) = \omega/\zeta + \Phi$, which is positive by the third shape condition. At $q = 1$, the identity $\kappa = \frac{\varepsilon-1}{2}\underline{\widehat{\pi}}^2 - \rho\underline{\widehat{\pi}}$ and $\widehat{\pi}(1) = \underline{\widehat{\pi}} + \beta$ give

$$F(1) = \widehat{\pi}(1)\left[\frac{\rho - (\varepsilon-1)\widehat{\pi}(1)/2}{\beta\zeta} - (\phi_\pi - 1)\right].$$

Indeed, $\widehat{\pi}(1) = (1-b)\underline{\widehat{\pi}} < 0$ and $\beta = -b\underline{\widehat{\pi}}$ give the exact factorization

$$-\frac{b\zeta}{1-b}F(1) = \rho - \frac{\varepsilon-1}{2}(1-b)\underline{\widehat{\pi}} - \beta\zeta(\phi_\pi - 1).$$

Thus, the last inequality in (A.2) is exactly equivalent to $F(1) < 0$. By continuity, F has a smallest zero $\bar{q} \in (0,1)$ and is positive on $(0,\bar{q})$, so $\sigma^s(q) > 0$ there. For $0 < q < 1$, $\widehat{\pi}(q) < \widehat{\pi}(1) < 0$.

As $\zeta \downarrow 0$ with b fixed, $\zeta\Phi$, $\zeta^2\Phi$ and $\beta\zeta(\phi_\pi - 1)$ vanish, because $\beta = -b\underline{\widehat{\pi}}$ and Φ do not depend on ζ, while ω and $\rho - \frac{\varepsilon-1}{2}(1-b)\underline{\widehat{\pi}}$ are positive. The shape conditions therefore hold for all sufficiently small ζ. Differentiation of (A.4) gives

$$F'(q) = -\frac{\kappa(1 + 1/\eta - \zeta)}{\beta\zeta}q^{1/\eta-\zeta} - \frac{(\varepsilon-1)\beta}{2}q^{\zeta-1} - \phi_y - (\phi_\pi - 1)\beta\zeta q^{\zeta-1}. \tag{A.5}$$

The first two terms are strictly negative, because $\zeta < 1$. Now let $\phi_y \geq 0$ and $\phi_\pi \geq 1$, not both with equality. Then the last two terms are nonpositive, so $F' < 0$ on $(0,1)$, and $\bar{q}$ is the unique zero of F in $(0,1)$. For $0 < q < 1$, $J(q) < 0$ because $\phi_y(q-1) \leq 0$ and $(\phi_\pi - 1)\widehat{\pi}(q) \leq 0$, with at least one inequality strict. □

We also impose the endpoint conditions

$$F'(\bar{q}) < 0, \qquad J(\bar{q}) < 0. \tag{A.6}$$

The first makes excess volatility vanish linearly at $\bar{q}$. The second requires the rule to set the real rate below the natural rate at $\bar{q}$, where excess volatility vanishes, so that the drift of the output ratio points into the interval. By Lemma A.1, both hold for every shape pair whenever $\phi_y \geq 0$ and $\phi_\pi \geq 1$, not both with equality. For given (b,ζ), all the conditions are strict inequalities and $\bar{q}$ is a simple zero of F, so they continue to hold under small changes in the policy coefficients. The next lemma shows that, for small ζ, the Taylor principle suffices.

Lemma A.2 (Existence under the Taylor principle)**.** *Let the rule satisfy the Taylor principle in equation (3.5). Then there is $\underline{b} \in [0,1)$ such that, for each $b \in (\underline{b},1)$, the pair (b,ζ) satisfies (A.2) and (A.6) for all sufficiently small $\zeta > 0$. If $\phi_y \geq 0$, one can take $\underline{b} = 0$.*

Proof. For $\underline{\widehat{\pi}} < \widehat{\pi} \leq 0$, let

$$q^{\mathrm{LR}}(\widehat{\pi}) = \left[1 + \frac{\rho}{\kappa}\widehat{\pi} - \frac{\varepsilon - 1}{2\kappa}\widehat{\pi}^2\right]^{\eta/(1+\eta)}$$

be the output ratio at which the drift of the Phillips curve in (3.3b) vanishes at constant centered inflation $\widehat{\pi}$. The function in the bracket is a concave quadratic that vanishes at $\underline{\widehat{\pi}}$, equals one at zero and increases in between. Since $\eta/(1+\eta) \in (0,1)$, q^{LR} is positive, increasing and concave, with $q^{\mathrm{LR}}(0) = 1$ and slope $\rho/[(1+1/\eta)\kappa]$ at zero.

Fix $b \in (0,1)$, let $\widehat{\pi}_b = (1-b)\underline{\widehat{\pi}} = \underline{\widehat{\pi}} + \beta < 0$ and $q_b = q^{\mathrm{LR}}(\widehat{\pi}_b) \in (0,1)$. By Lemma A.1, (A.2) holds for all small ζ. The identities $\kappa = \frac{\varepsilon-1}{2}\underline{\widehat{\pi}}^2 - \rho\underline{\widehat{\pi}}$ and $\omega = \rho - (\varepsilon-1)\underline{\widehat{\pi}}$ give $\beta\omega - \frac{\varepsilon-1}{2}\beta^2 = \kappa + \rho\widehat{\pi}_b - \frac{\varepsilon-1}{2}\widehat{\pi}_b^2 = \kappa q_b^{1+1/\eta}$, so (A.4) can be written as

$$\zeta F(q) = \frac{\kappa}{\beta}\left(q_b^{1+1/\eta} - q^{1+1/\eta-\zeta}\right) + \frac{(\varepsilon-1)\beta}{2}\left(1 - q^{\zeta}\right) - \zeta J(q).$$

On $(0,1)$, $\underline{\widehat{\pi}} < \widehat{\pi}(q) < 0$, so $|J(q)| \leq \bar{J} = |\phi_y| + |\phi_\pi - 1||\underline{\widehat{\pi}}|$. As $\zeta \downarrow 0$, $\zeta F(q)$ and $\zeta F'(q)$ therefore converge, uniformly on compact subsets of $(0,1]$, to $(\kappa/\beta)(q_b^{1+1/\eta} - q^{1+1/\eta})$ and $-(\kappa/\beta)(1+1/\eta)q^{1/\eta}$. Near zero, $q^{1+1/\eta-\zeta} \leq q^{1/\eta}$ gives $\zeta F(q) \geq (\kappa/\beta)(q_b^{1+1/\eta} - q^{1/\eta}) - \zeta\bar{J}$, which is positive for small q, uniformly in small ζ. Fix $0 < \epsilon < \min\{q_b, 1-q_b\}$. For all sufficiently small ζ, $F > 0$ on $(0, q_b - \epsilon]$, $F(q_b + \epsilon) < 0$ and $F' < 0$ on $[q_b - \epsilon, q_b + \epsilon]$. Hence $\bar{q} \in (q_b - \epsilon, q_b + \epsilon)$ and $F'(\bar{q}) < 0$. Since ϵ is arbitrary, $\bar{q} \to q_b$, and because $q^{\zeta} \to 1$ uniformly on $[q_b - \epsilon, 1]$,

$$J(\bar{q}) \longrightarrow J_b = \phi_y(q_b - 1) + (\phi_\pi - 1)\widehat{\pi}_b.$$

It remains to show $J_b < 0$. As $b \uparrow 1$, $\widehat{\pi}_b \uparrow 0$ and $(q_b - 1)/\widehat{\pi}_b \to \rho/[(1+1/\eta)\kappa]$, so

$$\frac{J_b}{\widehat{\pi}_b} \longrightarrow \phi_\pi - 1 + \frac{\rho\phi_y}{(1+1/\eta)\kappa} > 0$$

by the Taylor principle. Since $\widehat{\pi}_b < 0$, $J_b < 0$ for all b close to one. If $\phi_y \geq 0$, concavity gives $q_b - 1 \leq \rho\widehat{\pi}_b/[(1+1/\eta)\kappa]$ for every b, and hence

$$J_b \leq \widehat{\pi}_b\left[\phi_\pi - 1 + \frac{\rho\phi_y}{(1+1/\eta)\kappa}\right] < 0.$$

In either case $J(\bar{q}) < 0$ for all sufficiently small ζ. □

As $\zeta \downarrow 0$, the graph flattens at $\widehat{\pi}_b$, and the upper end of the state interval converges to the point $(q_b, \widehat{\pi}_b)$ of the nonlinear long-run Phillips curve. The endpoint condition then asks the rule to set the real rate below the natural rate at that point. Near the target, the long-run curve has the slope of its linearization in Appendix C.1, so this is the Taylor principle. When $\phi_y \geq 0$, concavity extends the sign to the entire below-target branch. When $\phi_y = 0$, the principle is also

necessary for the endpoint conditions: if $\phi_\pi \leq 1$, then $J(q) = (\phi_\pi - 1)\widehat{\pi}(q) \geq 0$ on $(0,1)$, and they fail for every shape pair. When $\phi_y > 0$, strict concavity gives $J_b < 0$ for every b even at the boundary of the principle, so by strictness the conditions also hold for some rules that violate it.

On $(0, \bar{q})$, F is positive, and excess volatility is the positive root $\sigma^s(q)$ in equation (3.6). It is positive in the interior and zero at $\bar{q}$, and it is strictly decreasing wherever F is, in particular on all of $(0, \bar{q})$ when $\phi_y \geq 0$ and $\phi_\pi \geq 1$, not both with equality.[3] Indeed,

$$\sigma\sigma^s(q) + \frac{1+\zeta}{2}(\sigma^s(q))^2 = F(q), \qquad (\sigma^s)'(q) = \frac{F'(q)}{\sigma + (1+\zeta)\sigma^s(q)}. \tag{A.7}$$

On this interval, define the state drift μ, the diffusion loading υ and the candidate state process by

$$\mu(q) = q\left[J(q) + \sigma\sigma^s(q) + (\sigma^s(q))^2\right], \qquad \upsilon(q) = q\sigma^s(q), \tag{A.8}$$

$$dq_t = \mu(q_t)dt + \upsilon(q_t)dZ_t, \qquad \widehat{\pi}_t = \widehat{\pi}(q_t), \quad \sigma^s_t = \sigma^s(q_t). \tag{A.9}$$

Along the graph, the policy rule in (2.20) sets $r_t - r^n = J(q_t)$, so (A.9) is the IS equation in (3.3a) with $\sigma^s_t = \sigma^s(q_t)$. Here υ denotes the diffusion loading of the ratio itself, distinct from the loading σ^s of its logarithm and from the cash velocity v of the main paper. For a twice continuously differentiable test function f, the generator is

$$\mathcal{L}f(q) = \mu(q)f'(q) + \frac{1}{2}\upsilon(q)^2 f''(q).$$

Below, $\mathbb{E}_q$ and $\mathbb{P}_q$ denote expectation and probability from a fixed initial value $q_0 = q$.

A.3 Drift matching and the proof of Lemma 2

Lemma A.3 (Drift matching)**.** *Let $q_t > 0$ satisfy the IS equation in (3.3a) with loading σ^s_t, and let $\widehat{\pi}_t = \widehat{\pi}(q_t)$. Then the Phillips curve in (3.3b) holds if and only if*

$$\sigma^\pi_t = \beta\zeta q_t^\zeta \sigma^s_t, \qquad \sigma\sigma^s_t + \frac{1+\zeta}{2}(\sigma^s_t)^2 = F(q_t). \tag{A.10}$$

In particular, any interior solution of (A.9) satisfies both limiting equations.

Proof. On the graph, the IS drift of q_t is $q_t[J(q_t) + \sigma^s_t(\sigma + \sigma^s_t)]$ and its loading is $q_t\sigma^s_t$. Apply

[3] Aggregate volatility is then countercyclical, as in Lee and Dordal i Carreras (2026).

Itô's formula to $\widehat{\pi}(q_t)$. Since $\widehat{\pi}'(q) = \beta\zeta q^{\zeta-1}$ and $\widehat{\pi}''(q) = \beta\zeta(\zeta-1)q^{\zeta-2}$,

$$\begin{aligned} d\widehat{\pi}(q_t) &= \left[\beta\zeta q_t^{\zeta-1} q_t \left(J(q_t) + \sigma\sigma_t^s + (\sigma_t^s)^2\right) + \frac{\beta\zeta(\zeta-1)}{2} q_t^{\zeta-2} q_t^2 (\sigma_t^s)^2\right] dt + \beta\zeta q_t^{\zeta} \sigma_t^s dZ_t \\ &= \beta\zeta q_t^{\zeta}\left[J(q_t) + \sigma\sigma_t^s + \frac{1+\zeta}{2}(\sigma_t^s)^2\right] dt + \beta\zeta q_t^{\zeta}\sigma_t^s dZ_t. \end{aligned} \tag{A.11}$$

By uniqueness of the semimartingale decomposition, the Phillips curve holds if and only if $\sigma_t^{\pi} = \beta\zeta q_t^{\zeta}\sigma_t^s$ and the drifts agree. The drift of the Phillips curve evaluated on the graph is

$$\begin{aligned} &\rho\widehat{\pi}(q) - \kappa\left(q^{1+1/\eta} - 1\right) - \frac{\varepsilon-1}{2}\widehat{\pi}(q)^2 \\ &\qquad = \omega\beta q^{\zeta} - \kappa q^{1+1/\eta} - \frac{\varepsilon-1}{2}\beta^2 q^{2\zeta} = \beta\zeta q^{\zeta}[F(q) + J(q)]. \end{aligned} \tag{A.12}$$

The constant term vanishes by the equation for $\underline{\widehat{\pi}}$, and the linear term uses $\rho - (\varepsilon-1)\underline{\widehat{\pi}} = \omega$. Since $\beta\zeta q^{\zeta} > 0$, the drifts agree if and only if the second equation in (A.10) holds. For an interior solution of (A.9), $\sigma_t^s = \sigma^s(q_t)$ is the positive root of that equation by (A.7), while (A.8) is exactly the IS drift and loading on the graph. □

The construction is simple: the volatility function $\sigma^s(q)$ is chosen so that the drift of the power graph $\widehat{\pi}(q)$ satisfies the nonlinear Phillips curve. In the IS equation, precautionary saving then offsets the effect of the real-rate deviation $J(q)$ on consumption growth. Excess volatility $\sigma^s(q)$ is the diffusion loading of the log output ratio. The loadings of the ratio itself and of centered inflation are $\upsilon(q) = q\sigma^s(q)$ and $\beta\zeta q^{\zeta}\sigma^s(q)$. Both vanish as $q \downarrow 0$, while excess volatility tends to a positive finite limit.

Proof of Lemma 2. Since (3.5) includes the Taylor principle, Lemma A.2 gives the existence of a pair. By Proposition A.1, the state equation (A.9) has a unique strong solution from q_0, which remains in $(0, \bar{q})$. With $\widehat{\pi}_t = \widehat{\pi}(q_t)$ and $\sigma_t^s = \sigma^s(q_t)$, the drift and loading of (A.9) are those of (3.3a), so this is the unique strong solution of the IS equation. By Lemma A.1, $\sigma^s(q_t)$ is the positive root of the second equation in (A.10), and Lemma A.3 gives the Phillips curve with $\sigma_t^{\pi} = \beta\zeta q_t^{\zeta}\sigma^s(q_t)$. Property (i) follows from $q_t \in (0, \bar{q})$ and $\bar{q} < 1$. Property (ii) follows because $\widehat{\pi}$ is strictly increasing, with $\widehat{\pi}(\bar{q}) < \widehat{\pi}(1) = (1-b)\underline{\widehat{\pi}} < 0$. Property (iii) follows because $\sigma^s > 0$ on $(0, \bar{q})$ and $\beta\zeta q_t^{\zeta} > 0$. Property (iv) restates the invariant-law part of Proposition A.1 and the comparison across shape pairs in Section A.6. □

A.4 The invariant distribution

For a fixed interior reference point $q_\circ \in (0, \bar{q})$, define the integrated drift-to-variance ratio $\mathcal{R}$ and the normalization $\mathcal{N}$ by

$$\mathcal{R}(q) = \int_{q_\circ}^{q} \frac{2\mu(y)}{\upsilon(y)^2}\, dy, \qquad \mathcal{N} = \int_0^{\bar{q}} \frac{e^{\mathcal{R}(q)}}{\upsilon(q)^2}\, dq. \tag{A.13}$$

Dividing by $\mathcal{N}$ removes the dependence of the density below on the arbitrary reference point $q_\circ$.

Proposition A.1 (Global state process and invariant law)**.** *For every $q_0 \in (0, \bar{q})$, (A.9) has a unique non-explosive strong solution defined for all time, and neither endpoint, 0 or $\bar{q}$, is reached from the interior. The endpoint zero is natural and the endpoint $\bar{q}$ is entrance. The normalization satisfies $0 < \mathcal{N} < \infty$, and the process has the unique invariant density*

$$\varphi_\star(q) = \frac{e^{\mathcal{R}(q)}}{\mathcal{N}\upsilon(q)^2}, \qquad 0 < q < \bar{q}. \tag{A.14}$$

This density has full interior support $(0, \bar{q})$. From every interior initial state, the law of q_t converges to it in total variation. Moreover, $\int_0^{\bar{q}} |\log q| \varphi_\star(q) dq < \infty$. Under invariant initialization, the output gap, centered inflation and excess volatility are nondegenerate and jointly stationary.

Section A.5 proves the proposition, using a Lyapunov drift bound and minorization on compact sets (Hairer and Mattingly, 2011). The scale and speed identities, the Lipschitz extension of the coefficients, and the Girsanov comparison used in that proof are provided separately in Sections A.9, A.10 and A.11.

Section A.5 shows that near $q = 0$ the drift of $\log q$ is positive.[4] Near $\bar{q}$, the diffusion loading υ vanishes linearly in the distance to $\bar{q}$ and the drift points inward. We use these endpoint rates to classify the boundaries, and to prove that the process never exits and that invariant mass is finite. Finally, we prove that the process converges in total variation to the invariant density.

For an integrable function f, write

$$\mathbb{E}_\star[f(q)] = \int_0^{\bar{q}} f(q)\varphi_\star(q) dq.$$

Inside such an expectation, the output ratio q, the gap $q - 1$, centered inflation $\widehat{\pi}(q)$ and excess volatility $\sigma^s(q)$ are random variables with the invariant law. Section A.6 discusses invariant initialization.

[4] Itô's formula applied to (A.9) gives this drift as $\ell(q) = J(q) + \sigma\sigma^s(q) + \frac{1}{2}(\sigma^s(q))^2$.

A.5 Proof of Proposition A.1

Proposition A.1 asserts three things about the diffusion (A.9): a unique strong solution exists for all time and never reaches either endpoint; the process has the invariant density (A.14); and the distribution converges to that density from every interior start. The proof proceeds in nine steps.

The first four concern a neighborhood of $q = 0$. A path cannot reach the origin in finite time. Occupation near the origin is still integrable, so the contribution of $\log q$ to consumption utility has a finite invariant mean. An interior reference point is reached in finite expected time from its left. A path that starts arbitrarily close to the origin does not reach that reference point in finite expected time, so the origin is not an entrance boundary. The next three steps ask the same questions at $q = \bar{q}$. The upper endpoint is inaccessible from the interior, the invariant mass near it is finite, and it is an entrance boundary: a path starting arbitrarily close to $\bar{q}$ reaches the interior in finite expected time.

Those seven steps evaluate two standard integrals of a one-dimensional diffusion, derived and explained in Section A.9. Scale distance decides whether one point is hit before another. Speed mass decides how much time is spent near an endpoint, and hence whether that tail contributes a finite amount to masses. A nested integral of the two decides whether the process can enter from an endpoint it cannot hit.

The eighth step extends the same coefficients Lipschitz-continuously up to both endpoints, so the state equation has a strong solution up to the first exit and needs no modification near the origin. That extension is provided in Section A.10. The ninth step produces the pull toward the interior and the overlap of transition probabilities that turn non-exit into convergence. The change of coordinate and the Girsanov weight used for the overlap are in Section A.11; the overlap constant itself is assembled in this proof.

Proof of Proposition A.1. The coefficients μ and υ are smooth with positive diffusion in the interior, giving local strong existence and uniqueness up to exit.

Let $\ell(q) = \mu(q)/q - (\sigma^s(q))^2/2$ denote the drift of $\log q$. Itô's formula on $\log q$ and (A.9) yield

$$d\log q_t = \ell(q_t)dt + \sigma^s(q_t)dZ_t, \qquad \ell(q) = J(q) + \sigma\sigma^s(q) + \frac{1}{2}(\sigma^s(q))^2. \tag{A.15}$$

The lower limit $\ell_0 = \lim_{q\downarrow 0}\ell(q)$ is obtained by evaluating each term at $q = 0+$. Centered inflation tends to $\widehat{\underline{\pi}}$, so

$$J(0+) = \phi_y(0-1) + (\phi_\pi - 1)\widehat{\underline{\pi}} = -\Phi$$

by the definition of Φ in (A.1). In (A.4) the two power terms vanish as $q \downarrow 0$, because $\zeta > 0$ and $1 + 1/\eta - \zeta > 0$, and therefore $F(0+) = \omega/\zeta + \Phi$, which is positive by the third shape condition.

Substituting this into the volatility root in equation (3.6) produces $\bar{\sigma}^s$ in (A.16) below. Hence

$$\ell_0 = -\Phi + \sigma\bar{\sigma}^s + \frac{1}{2}(\bar{\sigma}^s)^2.$$

Rationalizing the formula for $\bar{\sigma}^s$ gives $(1+\zeta)\bar{\sigma}^s + \sigma = \sqrt{\sigma^2 + 2(1+\zeta)(\omega/\zeta + \Phi)}$. Squaring and cancelling σ^2 yield

$$\sigma\bar{\sigma}^s + \frac{1+\zeta}{2}(\bar{\sigma}^s)^2 = \frac{\omega}{\zeta} + \Phi.$$

Subtract $\Phi + (\zeta/2)(\bar{\sigma}^s)^2$ from both sides:

$$\ell_0 = \frac{\omega}{\zeta} - \frac{\zeta}{2}(\bar{\sigma}^s)^2.$$

Solve the same identity for $(\bar{\sigma}^s)^2$ and substitute once more:

$$\ell_0 = \frac{\omega - \zeta^2\Phi}{\zeta(1+\zeta)} + \frac{\zeta\sigma\bar{\sigma}^s}{1+\zeta}.$$

The condition $\zeta^2\Phi < \omega$ in (A.2) and $\bar{\sigma}^s > 0$ make ℓ_0 strictly positive. Collected,

$$\bar{\sigma}^s = \frac{\sqrt{\sigma^2 + 2(1+\zeta)(\omega/\zeta + \Phi)} - \sigma}{1+\zeta} > 0, \tag{A.16}$$

$$\ell_0 = \frac{\omega - \zeta^2\Phi}{\zeta(1+\zeta)} + \frac{\zeta\sigma\bar{\sigma}^s}{1+\zeta} > 0. \tag{A.17}$$

Roughly speaking, the positive limit ℓ_0 is what makes the origin repelling. To turn it into the four conclusions listed above, form the scale integrand of Section A.9. From (A.8), $\upsilon(q) = q\sigma^s(q)$ and $\mu(q)/q = \ell(q) + (\sigma^s(q))^2/2$, so the scale integrand is

$$\frac{2\mu(q)}{\upsilon(q)^2} = \frac{2}{q(\sigma^s(q))^2}\left[\ell(q) + \frac{1}{2}(\sigma^s(q))^2\right].$$

Send $q \downarrow 0$. The bracket tends to $\ell_0 + (\bar{\sigma}^s)^2/2$ and the remaining coefficient times q tends to $2/(\bar{\sigma}^s)^2$. Therefore

$$\frac{2\mu(q)}{\upsilon(q)^2} \sim \frac{2\ell_0 + (\bar{\sigma}^s)^2}{q(\bar{\sigma}^s)^2} = \frac{1+\chi_0}{q}, \qquad \chi_0 \equiv \frac{2\ell_0}{(\bar{\sigma}^s)^2} > 0. \tag{A.18}$$

The exponent χ_0 is exactly the extra unit of $1/q$ coming from the positive limit of the drift of $\log q$. From the graph and (A.1),

$$J(q) + \Phi = \phi_y q + (\phi_\pi - 1)\beta q^\zeta.$$

Since $0 < \zeta < 1$, this is $O(q^\zeta)$. The two power terms in (A.4) are of orders $q^{1+1/\eta-\zeta}$ and q^ζ, with

$1 + 1/\eta - \zeta > 0$ because $\zeta < 1$. Also $J(q) - J(0+) = J(q) + \Phi$. Therefore

$$F(q) - F(0+) = O(q^{\vartheta}), \qquad \vartheta = \min(\zeta, 1 + 1/\eta - \zeta) > 0.$$

Differentiating the volatility root with respect to F, we obtain

$$\frac{d\sigma^s}{dF} = \frac{1}{\sigma + (1+\zeta)\sigma^s}.$$

On $(0, \bar{q})$, $F > 0$, hence $\sigma^s > 0$. With $\sigma > 0$, $\sigma + (1+\zeta)\sigma^s \geq \sigma$ at every q, including the limit $q \downarrow 0$. Thus the derivative is bounded by $1/\sigma$, uniformly in q and in F. By the mean value theorem, there is an intermediate value $F_\star(q)$ between $F(q)$ and $F(0+)$ such that[5]

$$\sigma^s(q) - \bar{\sigma}^s = \frac{F(q) - F(0+)}{\sigma + (1+\zeta)\sigma^s\big|_{F_\star(q)}}, \qquad |\sigma^s(q) - \bar{\sigma}^s| \leq \frac{1}{\sigma}|F(q) - F(0+)|,$$

where $\sigma^s|_{F_\star(q)}$ denotes the volatility root evaluated at $F_\star(q)$. The numerator is $O(q^{\vartheta})$, so $\sigma^s(q) - \bar{\sigma}^s = O(q^{\vartheta})$.

The log-ratio drift is $\ell = J + \sigma\sigma^s + (\sigma^s)^2/2$. Subtract the limit $\ell_0 = -\Phi + \sigma\bar{\sigma}^s + (\bar{\sigma}^s)^2/2$:

$$\ell(q) - \ell_0 = (J(q) + \Phi) + \sigma\big(\sigma^s(q) - \bar{\sigma}^s\big) + \frac{1}{2}\big(\sigma^s(q) - \bar{\sigma}^s\big)\big(\sigma^s(q) + \bar{\sigma}^s\big).$$

The first term is $O(q^{\zeta})$ and $\vartheta \leq \zeta$, so it is $O(q^{\vartheta})$. The remaining factors σ and $\sigma^s + \bar{\sigma}^s$ stay bounded near $q = 0$. Hence $\ell(q) - \ell_0 = O(q^{\vartheta})$.

We now return to scale. From $\mu/q = \ell + (\sigma^s)^2/2$ and $\upsilon = q\sigma^s$,

$$\frac{2\mu}{\upsilon^2} = \frac{2\ell + (\sigma^s)^2}{q(\sigma^s)^2}, \qquad \frac{1+\chi_0}{q} = \frac{2\ell_0 + (\bar{\sigma}^s)^2}{q(\bar{\sigma}^s)^2}.$$

Subtract:

$$\frac{2\mu(q)}{\upsilon(q)^2} - \frac{1+\chi_0}{q} = \frac{1}{q}\left(\frac{2\ell(q) + (\sigma^s(q))^2}{(\sigma^s(q))^2} - \frac{2\ell_0 + (\bar{\sigma}^s)^2}{(\bar{\sigma}^s)^2}\right).$$

The parenthesis is the increment of $G(l, s) = (2l + s^2)/s^2 = 2l/s^2 + 1$ between $(\ell(q), \sigma^s(q))$ and $(\ell_0, \bar{\sigma}^s)$. Rewriting gives

$$G(\ell, \sigma^s) - G(\ell_0, \bar{\sigma}^s) = \frac{2\big[(\ell - \ell_0)(\bar{\sigma}^s)^2 + \ell_0(\bar{\sigma}^s - \sigma^s)(\bar{\sigma}^s + \sigma^s)\big]}{(\sigma^s)^2(\bar{\sigma}^s)^2}.$$

The two differences in the numerator are $O(q^{\vartheta})$. The factors ℓ_0, $\bar{\sigma}^s$ and $\bar{\sigma}^s + \sigma^s$ stay bounded near $q = 0$. As $\sigma^s(q) \to \bar{\sigma}^s > 0$, $\sigma^s(q) \geq \bar{\sigma}^s/2$ for all sufficiently small q, so the denominator is bounded below by a positive constant that does not depend on q.[6] Hence the parenthesis is

[5] The intermediate point $F_\star(q)$ depends on q, and so does the denominator evaluated there.

[6] The q-dependence of that denominator therefore changes only the implicit multiplicative constant.

$O(q^{\vartheta})$. Multiplication by $1/q$ produces $O(q^{\vartheta-1})$.

The scale density from (A.81) is

$$\mathfrak{s}'(q) = \exp\left[-\int_{q_\circ}^{q} \frac{2\mu(\tilde{q})}{\upsilon(\tilde{q})^2}\, d\tilde{q}\right],$$

with the lower limit equal to $q_\circ$ so that $\mathfrak{s}'(q_\circ) = 1$. The lower tail uses $0 < q < q_\circ$. Hence

$$\mathfrak{s}'(q) = \exp\left[\int_{q}^{q_\circ} \frac{2\mu(\tilde{q})}{\upsilon(\tilde{q})^2}\, d\tilde{q}\right].$$

Split the integrand into the limit term near $q = 0$ and a remainder:

$$\mathfrak{e}(\tilde{q}) = \frac{2\mu(\tilde{q})}{\upsilon(\tilde{q})^2} - \frac{1+\chi_0}{\tilde{q}}.$$

We know from the above order calculation that $\mathfrak{e}(\tilde{q}) = O(\tilde{q}^{\vartheta-1})$ near zero, and $\mathfrak{e}$ is continuous on $(0, q_\circ]$. Then $\int_0^{q_\circ} |\mathfrak{e}|$ converges, because $\vartheta - 1 > -1$. Write

$$\int_q^{q_\circ} \frac{2\mu}{\upsilon^2}\, d\tilde{q} = \int_q^{q_\circ} \frac{1+\chi_0}{\tilde{q}}\, d\tilde{q} + \int_q^{q_\circ} \mathfrak{e}(\tilde{q})\, d\tilde{q}.$$

The first integral is elementary:

$$\int_q^{q_\circ} \frac{1+\chi_0}{\tilde{q}}\, d\tilde{q} = (1+\chi_0)\log\frac{q_\circ}{q}.$$

The second integral tends to the finite value $\int_0^{q_\circ} \mathfrak{e}$ as $q \downarrow 0$, because $\int_0^q \mathfrak{e} \to 0$. Therefore

$$\mathfrak{s}'(q) = \exp\left((1+\chi_0)\log\frac{q_\circ}{q} + \int_q^{q_\circ} \mathfrak{e}\right) = \left(\frac{q_\circ}{q}\right)^{1+\chi_0} \exp\left(\int_q^{q_\circ} \mathfrak{e}(\tilde{q})\, d\tilde{q}\right).$$

Rewrite the exponential as $\exp(\int_0^{q_\circ} \mathfrak{e}) \exp(-\int_0^q \mathfrak{e})$. The first factor is a positive constant. The second tends to one as $q \downarrow 0$. Collecting the powers of q gives

$$\mathfrak{s}'(q) = C_{\mathfrak{s}} q^{-1-\chi_0} \exp\left(-\int_0^q \mathfrak{e}(\tilde{q})\, d\tilde{q}\right), \qquad C_{\mathfrak{s}} = q_\circ^{1+\chi_0} \exp\left(\int_0^{q_\circ} \mathfrak{e}(\tilde{q})\, d\tilde{q}\right) > 0. \tag{A.19}$$

The exponential term in (A.19) tends to one, so $\mathfrak{s}'(q) \sim C_{\mathfrak{s}} q^{-1-\chi_0}$ near $q = 0$.

Speed limits follow from (A.82). From (A.8), $\upsilon(q)^2 = q^2(\sigma^s(q))^2$. Thus

$$\mathfrak{m}(q) = \frac{2}{\upsilon(q)^2 \mathfrak{s}'(q)} = \frac{2}{q^2(\sigma^s(q))^2}\, C_{\mathfrak{s}}^{-1} q^{1+\chi_0} \exp\left(\int_0^q \mathfrak{e}\right).$$

As $q \downarrow 0$, $(\sigma^s(q))^2 \to (\bar{\sigma}^s)^2$ and the exponential tends to one, so

$$\mathfrak{m}(q) \sim C_{\mathfrak{m}} q^{\chi_0 - 1}, \qquad C_{\mathfrak{m}} = \frac{2}{(\bar{\sigma}^s)^2 C_{\mathfrak{s}}} > 0.$$

Collected, as $q \downarrow 0$,

$$\mathfrak{s}'(q) \sim C_{\mathfrak{s}} q^{-1-\chi_0}, \qquad \mathfrak{m}(q) \sim C_{\mathfrak{m}} q^{\chi_0 - 1}. \tag{A.20}$$

Inaccessibility of $q = 0$. The first conclusion at the origin is that an interior path cannot reach it. Section A.9 shows that this holds when the scale distance from an interior point to 0 is infinite. With the normalization $\mathfrak{s}(q_\circ) = 0$, that distance is the elementary integral

$$\mathfrak{s}(q_\circ) - \mathfrak{s}(q) = \int_q^{q_\circ} \mathfrak{s}'(\tilde{q})\, d\tilde{q}.$$

The equivalent $\mathfrak{s}'(\tilde{q}) \sim C_{\mathfrak{s}} \tilde{q}^{-1-\chi_0}$ and $\chi_0 > 0$ give

$$\int_q^{q_\circ} \mathfrak{s}'(\tilde{q})\, d\tilde{q} \sim C_{\mathfrak{s}} \int_q^{q_\circ} \tilde{q}^{-1-\chi_0}\, d\tilde{q} = \frac{C_{\mathfrak{s}}}{\chi_0}(q^{-\chi_0} - q_\circ^{-\chi_0}) \sim \frac{C_{\mathfrak{s}}}{\chi_0} q^{-\chi_0}. \tag{A.21}$$

Sending $q \downarrow 0$ sends this remaining scale to $+\infty$. Equivalently, $\int_0^{q_\circ} \mathfrak{s}' = \infty$ and $\mathfrak{s}(q) \to -\infty$. By the hitting formula (A.80), the probability of reaching a neighborhood of 0 before a fixed interior barrier vanishes. The left endpoint is therefore inaccessible from the interior.

Integrable consumption utility near $q = 0$. The path never reaches 0, but consumption utility must remain integrable near the origin. Flow utility contains $\log C_t = \log A_t + \log L^n + \log q_t$, and $\log q \to -\infty$ as $q \downarrow 0$.

On $[q_\circ, \bar{q})$ the $\log q$ is bounded, and only the left tail can destroy the moment. Section A.9 shows that the invariant density is proportional to speed,[7] so the required integral is $\int_0^{q_\circ} |\log q|\, \mathfrak{m}(q)\, dq$.

First, speed mass itself is $\int_0^{q_\circ} \mathfrak{m}(q)\, dq$. The equivalent $\mathfrak{m}(q) \sim C_{\mathfrak{m}} q^{\chi_0 - 1}$ and $\chi_0 > 0$ make $\chi_0 - 1 > -1$, so the integral converges. Finite left-end speed mass is the contribution of $(0, q_\circ)$ to $\mathcal{N} < \infty$: after normalization, that tail is a finite piece of invariant probability.

Because the invariant density is a normalization of speed, $\int |\log q|\, \mathfrak{m}$ is finite if and only if $\mathbb{E}_\star[|\log q|] < \infty$. The power $\chi_0 - 1$ that made speed mass finite also absorbs the logarithm. To see this, note first that $|\log q| = o(q^{-\epsilon})$ as $q \downarrow 0$ for any $\epsilon > 0$.[8] Restrict further to $\epsilon \in (0, \chi_0)$, so that $|\log q|\, \mathfrak{m}(q)$ remains integrable:

$$|\log q|\, \mathfrak{m}(q) \sim C_{\mathfrak{m}} (|\log q|\, q^{\epsilon}) q^{\chi_0 - 1 - \epsilon}.$$

[7] This uses that scale diverges at both $q = 0$ and $q = \bar{q}$; the latter is verified below.

[8] Fix $\epsilon > 0$ and set $y = -\log q$. Then $q \downarrow 0$ is $y \to +\infty$, $q^{-\epsilon} = e^{\epsilon y}$, and $|\log q| / q^{-\epsilon} = y / e^{\epsilon y} \to 0$, which proves $|\log q|\, q^{\epsilon} \to 0$, and $|\log q| = o(q^{-\epsilon})$.

The parenthesis tends to zero, so it is bounded near $q = 0$, and therefore

$$|\log q|\,\mathfrak{m}(q) = O(q^{\chi_0 - 1 - \epsilon})$$

with $\chi_0 - 1 - \epsilon > -1$. Therefore $\int_0^{q_\circ} |\log q|\,\mathfrak{m}(q)\,dq < \infty$.

Finite-time arrival from the left. Finite speed mass implies that time spent near 0 is integrable, and scale has already ruled out hitting 0 from $q_\circ$. We now go the other way and compute the expected time to hit $q_\circ$ from $q \in (0, q_\circ)$, to see whether a path that starts to the left of an interior point reaches that point in finite time. The limit of that expectation as $q \downarrow 0$ is discussed afterwards, where it decides whether the origin is an entrance. The expectation itself is the unique solution of the generator equation in the scale-speed form of Section A.9.

The process cannot be started at 0, because 0 is not in the state space $(0, \bar{q})$. We therefore place an auxiliary left barrier a with $0 < a < q < q_\circ$, compute the expected time to leave the bounded interval $(a, q_\circ)$ from the interior point q, and only afterwards send $a \downarrow 0$. Exit from $(a, q_\circ)$ means the first time the state equals a or equals $q_\circ$. Hitting a is a move left toward 0, and hitting $q_\circ$ is a move right, to the reference point. Writing $\mathfrak{h}_a \wedge \mathfrak{h}_{q_\circ}$ for that first exit time, where $\mathfrak{h}_a$ denotes the first time the state equals a, define

$$\mathfrak{w}_a(q) = \mathbb{E}_q[\mathfrak{h}_a \wedge \mathfrak{h}_{q_\circ}]. \tag{A.22}$$

If the path starts at a barrier, the exit time is zero, so $\mathfrak{w}_a(a) = \mathfrak{w}_a(q_\circ) = 0$.

The generator problem

$$\mathcal{L}\psi = -1 \quad \text{on } (a, q_\circ), \qquad \psi(a) = \psi(q_\circ) = 0$$

has a unique C^2 solution ψ, computed below, and that solution is the expected exit time. To see the second claim, apply (A.79) to ψ along the path and stop at $t \wedge \mathfrak{h}_a \wedge \mathfrak{h}_{q_\circ}$. On the compact interval $[a, q_\circ]$, the diffusion coefficient and ψ' are bounded, so the stochastic integral is a true martingale and has mean zero. Taking $\mathbb{E}_q$ therefore gives

$$\mathbb{E}_q\big[\psi(q_{t\wedge\mathfrak{h}_a\wedge\mathfrak{h}_{q_\circ}})\big] = \psi(q) + \mathbb{E}_q\left[\int_0^{t\wedge\mathfrak{h}_a\wedge\mathfrak{h}_{q_\circ}} \mathcal{L}\psi(q_s)\,ds\right] = \psi(q) - \mathbb{E}_q[t \wedge \mathfrak{h}_a \wedge \mathfrak{h}_{q_\circ}]. \tag{A.23}$$

Since ψ is bounded on $[a, q_\circ]$, $\mathbb{E}_q[t \wedge \mathfrak{h}_a \wedge \mathfrak{h}_{q_\circ}] \le 2 \sup|\psi|$ for every t, so the exit time has finite expectation and is finite almost surely. Letting $t \to \infty$, with bounded convergence on the left, monotone convergence on the right and $\psi = 0$ at the exit point, gives $\psi(q) = \mathfrak{w}_a(q)$.

Section A.9 shows that the scale-speed form gives $\mathcal{L}f = \mathfrak{m}^{-1}\partial_q(f'/\mathfrak{s}')$; see (A.83). Substituting

$f = \mathfrak{w}_a$ and $\mathcal{L}\mathfrak{w}_a = -1$ therefore gives $\mathfrak{m}^{-1}\partial_q(\mathfrak{w}_a'/\mathfrak{s}') = -1$, or

$$\partial_q\left(\frac{\mathfrak{w}_a'(q)}{\mathfrak{s}'(q)}\right) = -\mathfrak{m}(q).$$

Integrating from a to an interior point $x \in [a, q_\circ]$, we obtain

$$\frac{\mathfrak{w}_a'(x)}{\mathfrak{s}'(x)} - \frac{\mathfrak{w}_a'(a)}{\mathfrak{s}'(a)} = -\int_a^x \mathfrak{m}(y)\,dy.$$

Write K_a for the unknown constant $\mathfrak{w}_a'(a)/\mathfrak{s}'(a)$. Then

$$\mathfrak{w}_a'(x) = \mathfrak{s}'(x)\left(K_a - \int_a^x \mathfrak{m}(y)\,dy\right).$$

A second integration from a to q, using $\mathfrak{w}_a(a) = 0$, produces

$$\mathfrak{w}_a(q) = K_a\big(\mathfrak{s}(q) - \mathfrak{s}(a)\big) - \int_a^q \mathfrak{s}'(x)\left(\int_a^x \mathfrak{m}(y)\,dy\right)dx. \tag{A.24}$$

The remaining constant is fixed by $\mathfrak{w}_a(q_\circ) = 0$:

$$K_a = \frac{\displaystyle\int_a^{q_\circ} \mathfrak{s}'(x)\left(\int_a^x \mathfrak{m}(y)\,dy\right)dx}{\mathfrak{s}(q_\circ) - \mathfrak{s}(a)}.$$

Integration by parts rewrites the double integral as

$$\int_a^{q_\circ} \mathfrak{s}'(x)\left(\int_a^x \mathfrak{m}(y)\,dy\right)dx = \int_a^{q_\circ} \big(\mathfrak{s}(q_\circ) - \mathfrak{s}(y)\big)\mathfrak{m}(y)\,dy.$$

The same identity on $[a, q]$ replaces the second term in (A.24) by $\int_a^q (\mathfrak{s}(q)-\mathfrak{s}(y))\mathfrak{m}(y)\,dy$. Therefore

$$\mathfrak{w}_a(q) = \frac{\mathfrak{s}(q) - \mathfrak{s}(a)}{\mathfrak{s}(q_\circ) - \mathfrak{s}(a)} \int_a^{q_\circ} \big(\mathfrak{s}(q_\circ) - \mathfrak{s}(y)\big)\mathfrak{m}(y)\,dy - \int_a^q \big(\mathfrak{s}(q) - \mathfrak{s}(y)\big)\mathfrak{m}(y)\,dy. \tag{A.25}$$

The first integral still runs over the whole of $(a, q_\circ)$. Split it at the starting point $y = q$:

$$\int_a^{q_\circ} \big(\mathfrak{s}(q_\circ) - \mathfrak{s}(y)\big)\mathfrak{m}(y)\,dy = \int_a^q \big(\mathfrak{s}(q_\circ) - \mathfrak{s}(y)\big)\mathfrak{m}(y)\,dy + \int_q^{q_\circ} \big(\mathfrak{s}(q_\circ) - \mathfrak{s}(y)\big)\mathfrak{m}(y)\,dy.$$

Substitute this split into (A.25), and write $\mathfrak{p}_a(q) = (\mathfrak{s}(q)-\mathfrak{s}(a))/(\mathfrak{s}(q_\circ)-\mathfrak{s}(a))$ for the y-independent prefactor. Then

$$\begin{aligned}\mathfrak{w}_a(q) = \mathfrak{p}_a(q) \int_a^q \big(\mathfrak{s}(q_\circ) - \mathfrak{s}(y)\big)\mathfrak{m}(y)\,dy + \mathfrak{p}_a(q) \int_q^{q_\circ} \big(\mathfrak{s}(q_\circ) - \mathfrak{s}(y)\big)\mathfrak{m}(y)\,dy \\ - \int_a^q \big(\mathfrak{s}(q) - \mathfrak{s}(y)\big)\mathfrak{m}(y)\,dy,\end{aligned} \tag{A.26}$$

where the subtracted integral has upper limit q, not $q_\circ$. In (A.26), the only piece on $(q, q_\circ)$ is the middle integral, whose integrand is $\mathfrak{p}_a(q)(\mathfrak{s}(q_\circ) - \mathfrak{s}(y))\mathfrak{m}(y)$.

On (a, q) the first and third integrals are both present. The first supplies $\mathfrak{p}_a(q)(\mathfrak{s}(q_\circ) - \mathfrak{s}(y))\mathfrak{m}(y)$, and the third subtracts $(\mathfrak{s}(q) - \mathfrak{s}(y))\mathfrak{m}(y)$. The net coefficient of $\mathfrak{m}(y)$ is therefore

$$\begin{aligned}
&\frac{(\mathfrak{s}(q) - \mathfrak{s}(a))(\mathfrak{s}(q_\circ) - \mathfrak{s}(y))}{\mathfrak{s}(q_\circ) - \mathfrak{s}(a)} - (\mathfrak{s}(q) - \mathfrak{s}(y)) \\
&\quad = \frac{1}{\mathfrak{s}(q_\circ) - \mathfrak{s}(a)}\Big[(\mathfrak{s}(q) - \mathfrak{s}(a))(\mathfrak{s}(q_\circ) - \mathfrak{s}(y)) - (\mathfrak{s}(q) - \mathfrak{s}(y))(\mathfrak{s}(q_\circ) - \mathfrak{s}(a))\Big] \\
&\quad = \frac{(\mathfrak{s}(q_\circ) - \mathfrak{s}(q))(\mathfrak{s}(y) - \mathfrak{s}(a))}{\mathfrak{s}(q_\circ) - \mathfrak{s}(a)}.
\end{aligned}$$

The factor $\mathfrak{s}(q_\circ) - \mathfrak{s}(q)$ does not depend on y, so it leaves the integral. The contribution of (a, q) is therefore

$$\frac{\mathfrak{s}(q_\circ) - \mathfrak{s}(q)}{\mathfrak{s}(q_\circ) - \mathfrak{s}(a)} \int_a^q (\mathfrak{s}(y) - \mathfrak{s}(a))\mathfrak{m}(y)\, dy.$$

Adding the $(q, q_\circ)$ contribution already recorded gives

$$\begin{aligned}
\mathfrak{w}_a(q) = &\frac{\mathfrak{s}(q_\circ) - \mathfrak{s}(q)}{\mathfrak{s}(q_\circ) - \mathfrak{s}(a)} \int_a^q (\mathfrak{s}(y) - \mathfrak{s}(a))\mathfrak{m}(y)\, dy \\
&+ \frac{\mathfrak{s}(q) - \mathfrak{s}(a)}{\mathfrak{s}(q_\circ) - \mathfrak{s}(a)} \int_q^{q_\circ} (\mathfrak{s}(q_\circ) - \mathfrak{s}(y))\mathfrak{m}(y)\, dy. \qquad \text{(A.27)}
\end{aligned}$$

The two prefactors are the hitting probabilities from (A.80): $(\mathfrak{s}(q_\circ) - \mathfrak{s}(q))/(\mathfrak{s}(q_\circ) - \mathfrak{s}(a))$ is $\mathbb{P}_q(\mathfrak{h}_a < \mathfrak{h}_{q_\circ})$, and $\mathfrak{p}_a(q)$ is $\mathbb{P}_q(\mathfrak{h}_{q_\circ} < \mathfrak{h}_a)$. The first integral runs over states to the left of the start, the second over states to the right. Each integrand is remaining scale from that side's absorbing barrier, weighted by speed. The same computation on any interior interval (a, c) gives (A.27) with $q_\circ$ replaced by c.

Now send the auxiliary left barrier a to 0, still starting from the same interior q. We already know $\mathfrak{s}(a) \to -\infty$, which sends the probability of hitting a before $q_\circ$ to zero, so $\mathfrak{h}_a \wedge \mathfrak{h}_{q_\circ}$ increases to $\mathfrak{h}_{q_\circ}$. In the second prefactor $\mathfrak{p}_a(q)$, both the numerator and the denominator are dominated by $-\mathfrak{s}(a) \to +\infty$, so $\mathfrak{p}_a(q) \to 1$. The second integral does not depend on a, so the second term of $\mathfrak{w}_a(q)$ tends to $\int_q^{q_\circ} (\mathfrak{s}(q_\circ) - \mathfrak{s}(y))\mathfrak{m}(y)\, dy$.

The first term in (A.27) contains $\mathfrak{s}(a)$, so the limit $a \downarrow 0$ requires care. Write $\mathfrak{M}(z, q) = \int_z^q \mathfrak{m}(y)\, dy$ for the inner speed mass and apply integration by parts to the first integral:

$$\int_a^q (\mathfrak{s}(y) - \mathfrak{s}(a))\mathfrak{m}(y)\, dy = \int_a^q \mathfrak{s}'(z)\, \mathfrak{M}(z, q)\, dz.$$

The identity $\mathfrak{s}(q) - \mathfrak{s}(a) = \int_a^q \mathfrak{s}'(z)\, dz$ makes the ratio of these two integrals the $\mathfrak{s}'$-weighted

average of $\mathfrak{M}(\cdot, q)$ on (a, q). The first term of (A.27) is therefore

$$\frac{\mathfrak{s}(q_\circ) - \mathfrak{s}(q)}{\mathfrak{s}(q_\circ) - \mathfrak{s}(a)} \int_a^q \mathfrak{s}'(z)\, \mathfrak{M}(z, q)\, dz = (\mathfrak{s}(q_\circ) - \mathfrak{s}(q)) \frac{\int_a^q \mathfrak{s}'(z)\, \mathfrak{M}(z, q)\, dz}{\int_a^q \mathfrak{s}'(z)\, dz}\, \mathfrak{p}_a(q).$$

Finite left-end speed mass implies $\mathfrak{M}(z, q) \to \int_0^q \mathfrak{m}$ as $z \downarrow 0$, and $0 \le \mathfrak{M}(z, q) \le \int_0^q \mathfrak{m}$ for every $z \in (a, q)$. Because $\mathfrak{s}'$ is not integrable at 0,[9] the probability measure with density $\mathfrak{s}'(z)/\int_a^q \mathfrak{s}'$ on (a, q) puts vanishing mass on any interval $[z_0, q]$ with $z_0 \in (0, q)$ as $a \downarrow 0$. The weighted average thus sees only values of $\mathfrak{M}(z, q)$ with z near a, and tends to $\int_0^q \mathfrak{m}$. Together with $\mathfrak{p}_a(q) \to 1$, the first term tends to $(\mathfrak{s}(q_\circ) - \mathfrak{s}(q)) \int_0^q \mathfrak{m}$. After the limit, $\mathfrak{s}'$ is no longer present: it was the weighting density, not a surviving factor. The left side $\mathfrak{w}_a(q) = \mathbb{E}_q[\mathfrak{h}_a \wedge \mathfrak{h}_{q_\circ}]$ increases to $\mathbb{E}_q[\mathfrak{h}_{q_\circ}]$ by monotone convergence. Collecting the two limits yields

$$\mathbb{E}_q[\mathfrak{h}_{q_\circ}] = (\mathfrak{s}(q_\circ) - \mathfrak{s}(q)) \int_0^q \mathfrak{m}(y)\, dy + \int_q^{q_\circ} (\mathfrak{s}(q_\circ) - \mathfrak{s}(y)) \mathfrak{m}(y)\, dy. \tag{A.28}$$

Both pieces are finite for each fixed $q \in (0, q_\circ)$: left-end speed mass is finite, remaining scale from q to $q_\circ$ is finite, and the second integrand is continuous on the compact interval $[q, q_\circ]$. Thus $q_\circ$ is reached in finite expected time from every interior start to its left.

The origin is not an entrance boundary. The result above concerns the expected time to go from a fixed interior start q to the interior reference $q_\circ$. We now ask the opposite question: can a path that begins arbitrarily close to 0 reach $q_\circ$ in finite expected time? The process cannot be initialized at $q = 0$, so the question is the limit

$$\lim_{q \downarrow 0} \mathbb{E}_q[\mathfrak{h}_{q_\circ}]. \tag{A.29}$$

If it is finite, $q = 0$ is an entrance boundary: the state can come from 0 into the interior and hit $q_\circ$ in finite average time. If the limit is infinite, it cannot enter from 0 in finite time.[10]

Send $q \downarrow 0$ in (A.28). The first part is the product

$$(\mathfrak{s}(q_\circ) - \mathfrak{s}(q)) \int_0^q \mathfrak{m}(y)\, dy.$$

The remaining scale $\mathfrak{s}(q_\circ) - \mathfrak{s}(q) = \int_q^{q_\circ} \mathfrak{s}' \sim (C_\mathfrak{s}/\chi_0) q^{-\chi_0}$ was computed in (A.21). Finite speed mass near 0 and $\mathfrak{m}(y) \sim C_\mathfrak{m} y^{\chi_0 - 1}$ give

$$\int_0^q \mathfrak{m}(y)\, dy \sim C_\mathfrak{m} \int_0^q y^{\chi_0 - 1}\, dy = \frac{C_\mathfrak{m}}{\chi_0} q^{\chi_0}.$$

[9] In (A.20), $-1 - \chi_0 < -1$.

[10] This is not the earlier inaccessibility test, which asked whether a path starting at $q_\circ$ can hit 0.

The product of these equivalents is

$$\frac{C_{\mathfrak{s}}}{\chi_0} q^{-\chi_0} \cdot \frac{C_{\mathfrak{m}}}{\chi_0} q^{\chi_0} = \frac{C_{\mathfrak{s}} C_{\mathfrak{m}}}{\chi_0^2} = \frac{2}{\chi_0^2 (\bar{\sigma}^s)^2},$$

using $C_{\mathfrak{m}} = 2/((\bar{\sigma}^s)^2 C_{\mathfrak{s}})$. Thus the first piece in (A.28) tends to a finite positive constant and cannot decide whether the limit of $\mathbb{E}_q[\mathfrak{h}_{q_\circ}]$ is finite.

The second piece is $\int_q^{q_\circ} (\mathfrak{s}(q_\circ) - \mathfrak{s}(y)) \mathfrak{m}(y)\, dy$. The integrand is nonnegative. Sending $q \downarrow 0$ thus enlarges the domain of integration monotonically to $(0, q_\circ)$, and monotone convergence yields

$$\lim_{q \downarrow 0} \int_q^{q_\circ} (\mathfrak{s}(q_\circ) - \mathfrak{s}(y)) \mathfrak{m}(y)\, dy = \int_0^{q_\circ} (\mathfrak{s}(q_\circ) - \mathfrak{s}(y)) \mathfrak{m}(y)\, dy.$$

Using $\mathfrak{s}(q_\circ) - \mathfrak{s}(y) = \int_y^{q_\circ} \mathfrak{s}'$, the limit can be written as

$$\int_0^{q_\circ} \mathfrak{m}(y) \left(\int_y^{q_\circ} \mathfrak{s}'(\tilde{q})\, d\tilde{q} \right) dy. \tag{A.30}$$

The limit (A.29) is finite if and only if the nested integral (A.30) is finite, because the first piece stays bounded. If the nested integral diverges, then $\mathbb{E}_q[\mathfrak{h}_{q_\circ}] \to \infty$ as $q \downarrow 0$: starting closer to 0 makes the expected clock from the left endpoint to $q_\circ$ arbitrarily long.

The inner integral is the remaining scale already computed in (A.21), which is asymptotic to a multiple of $y^{-\chi_0}$. Multiplication by $\mathfrak{m}(y) \sim C_{\mathfrak{m}} y^{\chi_0 - 1}$ cancels χ_0 in the exponent:

$$\mathfrak{m}(y) \int_y^{q_\circ} \mathfrak{s}'(\tilde{q})\, d\tilde{q} \sim C_{\mathfrak{m}} \frac{C_{\mathfrak{s}}}{\chi_0} y^{\chi_0 - 1} y^{-\chi_0} = \frac{2}{\chi_0 (\bar{\sigma}^s)^2}\, y^{-1},$$

where the last equality uses $C_{\mathfrak{m}} = 2/((\bar{\sigma}^s)^2 C_{\mathfrak{s}})$. The integral $\int_0^{q_\circ} y^{-1}\, dy$ diverges logarithmically, so the nested integral is infinite and $\mathbb{E}_q[\mathfrak{h}_{q_\circ}] \to \infty$ as $q \downarrow 0$.

The origin is therefore inaccessible from the interior, and the expected time to enter from it is infinite. In the boundary classification of Helland (1996, Theorem 8), this pair of conclusions makes zero a natural boundary.[11] Finite left-end speed mass is compatible with it: occupation near 0 can be integrable while the entrance clock is not. We now show that at the upper endpoint the same nested integral converges, so that $\bar{q}$ is an entrance rather than a natural boundary.

Inaccessibility of $q = \bar{q}$. The same three questions are now asked at the upper end $q = \bar{q}$, the smallest zero of F in $(0, 1)$ from Lemma A.1. Near $\bar{q}$ we use the local coordinate $w = \bar{q} - q > 0$. As $q \uparrow \bar{q}$ we check, in order, that the scale distance from an interior point to $\bar{q}$ is infinite, that speed mass near $\bar{q}$ is finite, and that the nested entrance integral from $\bar{q}$ down to $q_\circ$ converges.

We begin with the volatility root. At $q = \bar{q}$, $F(\bar{q}) = 0$, so $\sigma^s(\bar{q}) = 0$. Rationalizing the

[11] In Lee and Dordal i Carreras (2026), the stabilized path is likewise a natural boundary of the martingale equilibria when fundamental volatility is positive.

numerator, we obtain

$$\sigma^s(q) = \frac{2F(q)}{\sqrt{\sigma^2 + 2(1+\zeta)F(q)} + \sigma}. \tag{A.31}$$

The mean value theorem supplies $q_\star \in (q, \bar{q})$ with $F(q) = F'(q_\star)(q - \bar{q}) = -F'(q_\star)(\bar{q} - q)$. As $q \uparrow \bar{q}$, the intermediate point $q_\star$ tends to $\bar{q}$, so $-F'(q_\star) \to -F'(\bar{q}) > 0$ by the endpoint condition (A.6), and the denominator tends to $\sigma + \sigma = 2\sigma$. Therefore

$$\sigma^s(q) \sim \frac{-F'(\bar{q})}{\sigma}(\bar{q} - q) = c_u(\bar{q} - q), \qquad c_u = -\frac{F'(\bar{q})}{\sigma} > 0.$$

The diffusion loading in (A.8) is $\upsilon(q) = q\sigma^s(q)$. Since $q \to \bar{q}$,

$$\upsilon(q) \sim \bar{q}c_u(\bar{q} - q). \tag{A.32}$$

The drift is $\mu(q) = q[J(q) + \sigma\sigma^s(q) + (\sigma^s(q))^2]$. Excess volatility vanishes at the root, and $q \to \bar{q}$, so

$$\mu(q) \longrightarrow \mu_u = \bar{q}J(\bar{q}). \tag{A.33}$$

The endpoint condition $J(\bar{q}) < 0$ and $\bar{q} \in (0, 1)$ give $\mu_u < 0$: near $\bar{q}$, the drift points back into the interval.

The scale integrand is $2\mu/\upsilon^2$. The functions F, J and the excess volatility in (A.31) are C^∞ near $\bar{q}$ inside $(0, 1)$, the last because its radicand equals $\sigma^2 > 0$ at $\bar{q}$. The condition $F(\bar{q}) = 0$ gives $F(q) = F'(\bar{q})(q-\bar{q})+O(w^2)$. With $c_u = -F'(\bar{q})/\sigma$ and $q-\bar{q} = -w$, we have $F(q) = \sigma c_u w+O(w^2)$, so $F(q) = O(w)$. The denominator in (A.31) therefore starts at 2σ:

$$\sqrt{\sigma^2 + 2(1+\zeta)F(q)} = \sigma\sqrt{1 + \frac{2(1+\zeta)F(q)}{\sigma^2}} = \sigma + \frac{(1+\zeta)F(q)}{\sigma} + O(w^2) = \sigma + O(w).$$

Adding σ gives $\sqrt{\sigma^2 + 2(1+\zeta)F} + \sigma = 2\sigma + O(w)$. At $w = 0$ this is exactly 2σ, and the error is of order w. Substitute into (A.31):

$$\sigma^s(q) = \frac{2(\sigma c_u w + O(w^2))}{2\sigma + O(w)} = c_u w + O(w^2).$$

With this, the loading is

$$\upsilon(q) = (\bar{q} - w)(c_u w + O(w^2)) = \bar{q}c_u w + O(w^2),$$

which is consistent with (A.32). Squaring gives $\upsilon(q)^2 = \bar{q}^2 c_u^2 w^2(1 + O(w))$. Also $J(q) =$

$J(\bar{q}) + O(w)$, $\sigma\sigma^s = O(w)$ and $(\sigma^s)^2 = O(w^2)$, so

$$\mu(q) = (\bar{q} - w)\big[J(\bar{q}) + O(w)\big] = \mu_u + O(w),$$

which is consistent with (A.33). Therefore

$$\frac{2\mu(q)}{\upsilon(q)^2} = \frac{2\big(\mu_u + O(w)\big)}{\bar{q}^2 c_u^2 w^2 (1 + O(w))} = -\frac{C_u}{w^2} + O(w^{-1}), \qquad C_u = -\frac{2\mu_u}{\bar{q}^2 c_u^2} > 0.$$

The next term of the expansion is also determined. Because σ^s is smooth with a simple zero at $\bar{q}$, $\upsilon(q) = w\, g(w)$ for a smooth function g with $g(0) = \bar{q}c_u \neq 0$. Hence $w^2 \cdot 2\mu/\upsilon^2 = 2\mu/g^2$ is a smooth function of w near zero, with value $-C_u$ at $w = 0$. The limit

$$\lambda_u = \lim_{q \uparrow \bar{q}} (\bar{q} - q) \left(\frac{2\mu(q)}{\upsilon(q)^2} + \frac{C_u}{(\bar{q} - q)^2} \right) = \lim_{w \downarrow 0} \frac{2\mu(q)\, w^2 + C_u \upsilon(q)^2}{\upsilon(q)^2 w}$$

therefore exists and is finite, and $2\mu/\upsilon^2$ near $\bar{q}$ can be written as

$$\frac{2\mu(q)}{\upsilon(q)^2} = -\frac{C_u}{(\bar{q} - q)^2} + \frac{\lambda_u}{\bar{q} - q} + e_u(q), \tag{A.34}$$

where $e_u(q)$ is bounded on a left neighborhood of $\bar{q}$.

For $q > q_\circ$ the scale density is

$$\mathfrak{s}'(q) = \exp\left[-\int_{q_\circ}^{q} \frac{2\mu(\tilde{q})}{\upsilon(\tilde{q})^2}\, d\tilde{q} \right].$$

Fix an interior point $q_1 \in (q_\circ, \bar{q})$ in the neighborhood where (A.34) holds. Because the integral from $q_\circ$ to q_1 is a constant, consider the integral from q_1 to q. On (q_1, q), substituting (A.34), we obtain

$$-\int_{q_1}^{q} \frac{2\mu}{\upsilon^2}\, d\tilde{q} = C_u \int_{q_1}^{q} \frac{d\tilde{q}}{(\bar{q} - \tilde{q})^2} - \lambda_u \int_{q_1}^{q} \frac{d\tilde{q}}{\bar{q} - \tilde{q}} - \int_{q_1}^{q} e_u(\tilde{q})\, d\tilde{q}.$$

The first integral is $1/(\bar{q} - q) - 1/(\bar{q} - q_1)$. The second is $-\log(\bar{q} - q) + \log(\bar{q} - q_1)$. The third converges to a finite limit as $q \uparrow \bar{q}$ because e_u is bounded. Therefore

$$-\int_{q_1}^{q} \frac{2\mu}{\upsilon^2}\, d\tilde{q} = \frac{C_u}{\bar{q} - q} + \lambda_u \log(\bar{q} - q) + K_u(q),$$

where $K_u(q)$ tends to a finite limit $K_u(\bar{q}-)$. Exponentiating gives

$$\mathfrak{s}'(q) = C_{\mathfrak{s},u}(q)\, (\bar{q} - q)^{\lambda_u} \exp\left[\frac{C_u}{\bar{q} - q} \right], \qquad C_{\mathfrak{s},u}(q) = \exp\left(-\int_{q_\circ}^{q_1} \frac{2\mu}{\upsilon^2}\, d\tilde{q} + K_u(q) \right),$$

where $C_{\mathfrak{s},u}(q)$ converges to a finite limit $C_{\mathfrak{s},u} > 0$ as $q \uparrow \bar{q}$.

With $y = 1/w$,

$$\int^{\bar{q}} \exp\left[\frac{C_u}{\bar{q}-q}\right] dq = \int_0 \exp\left[\frac{C_u}{w}\right] dw = \int^{\infty} e^{C_u y}\, y^{-2}\, dy = \infty,$$

and the exponential dominates the power $(\bar{q}-q)^{\lambda_u}$. Therefore $\int_{q_\circ}^{\bar{q}} \mathfrak{s}' = \infty$ and $\mathfrak{s}(q) \to +\infty$ as $q \uparrow \bar{q}$. By (A.80), the probability of hitting a neighborhood of $\bar{q}$ before a fixed interior barrier vanishes. The upper endpoint is inaccessible from the interior.

Finite invariant mass near $q = \bar{q}$. Infinite scale distance shows that $\bar{q}$ is not accessible from the interior. The contribution of a neighborhood of $\bar{q}$ to the normalizing constant $\mathcal{N}$ is the speed mass of that neighborhood, again by Section A.9. We show that this tail is finite. Speed is $\mathfrak{m} = 2/(\upsilon^2 \mathfrak{s}')$. Insert $\upsilon^2 \sim \bar{q}^2 c_u^2 (\bar{q}-q)^2$:

$$\mathfrak{m}(q) \sim C_{\mathfrak{m},u} (\bar{q}-q)^{-2-\lambda_u} \exp\left[-\frac{C_u}{\bar{q}-q}\right], \qquad C_{\mathfrak{m},u} = \frac{2}{\bar{q}^2 c_u^2 C_{\mathfrak{s},u}} > 0.$$

The exponential now decays as $q \uparrow \bar{q}$. With $w = \bar{q} - \tilde{q}$,

$$\int_q^{\bar{q}} \mathfrak{m}(\tilde{q})\, d\tilde{q} \sim C_{\mathfrak{m},u} \int_0^{\bar{q}-q} w^{-2-\lambda_u} \exp\left[-\frac{C_u}{w}\right] dw. \tag{A.35}$$

Differentiate $\Gamma(w) = w^{-\lambda_u} \exp(-C_u/w)$:

$$\Gamma'(w) = (C_u - \lambda_u w)\, w^{-2-\lambda_u} e^{-C_u/w}.$$

For small w, $C_u - \lambda_u w \neq 0$, so

$$w^{-2-\lambda_u} e^{-C_u/w} = \frac{\Gamma'(w)}{C_u - \lambda_u w} = \frac{\Gamma'(w)}{C_u} \cdot \frac{1}{1-(\lambda_u/C_u)w} = \frac{\Gamma'(w)}{C_u}(1+O(w)).$$

The integral in (A.35) from 0 to $\bar{w} = \bar{q} - q$ thus becomes

$$\int_0^{\bar{w}} w^{-2-\lambda_u} e^{-C_u/w}\, dw = \frac{1}{C_u}\int_0^{\bar{w}} \Gamma'(w)\, dw + \int_0^{\bar{w}} \Gamma'(w)\, O(w)\, dw.$$

The first integral is $\Gamma(\bar{w}) - \Gamma(0+)$. As $w \downarrow 0$, $\exp(-C_u/w)$ dominates every power of w, so $\Gamma(0+) = 0$. The second integral satisfies

$$\left|\int_0^{\bar{w}} \Gamma'(w)\, O(w)\, dw\right| \leq K \int_0^{\bar{w}} w\, \Gamma'(w)\, dw \leq K\bar{w} \int_0^{\bar{w}} \Gamma'(w)\, dw = K\bar{w}\,\Gamma(\bar{w})$$

for some finite K, because $\Gamma' > 0$ for all sufficiently small w. Therefore

$$\int_0^{\bar{w}} w^{-2-\lambda_u} e^{-C_u/w}\, dw = \frac{\Gamma(\bar{w})}{C_u}(1+O(\bar{w})) = \frac{\bar{w}^{-\lambda_u}}{C_u} \exp\left(-\frac{C_u}{\bar{w}}\right)(1+O(\bar{w})).$$

The upper limit $\bar{w} = \bar{q} - q$ therefore gives

$$\int_q^{\bar{q}} \mathfrak{m}(\tilde{q})\, d\tilde{q} \sim C_{u,\star}(\bar{q} - q)^{-\lambda_u} \exp\left[-\frac{C_u}{\bar{q} - q}\right], \qquad C_{u,\star} = \frac{C_{\mathfrak{m},u}}{C_u} > 0. \tag{A.36}$$

In particular the tail is finite for every $q < \bar{q}$, so the upper-end contribution to $\mathcal{N}$ is finite. Together with finite left-end speed mass, this proves $0 < \mathcal{N} < \infty$.

The upper endpoint is an entrance boundary. Inaccessibility and finite speed mass still leave open whether a path can be thought of as entering from $\bar{q}$. As for the origin, the entrance clock is the expected time to reach the interior reference $q_\circ$, now starting from $q \in (q_\circ, \bar{q})$, and then the limit of that expectation as $q \uparrow \bar{q}$. A finite limit means that $\bar{q}$ is an entrance. Using (A.27) on the interval $(q_\circ, c)$ with $c \in (q, \bar{q})$, the expected exit time from q is

$$\mathbb{E}_q[\mathfrak{h}_{q_\circ} \wedge \mathfrak{h}_c] = \frac{\mathfrak{s}(c) - \mathfrak{s}(q)}{\mathfrak{s}(c) - \mathfrak{s}(q_\circ)} \int_{q_\circ}^{q} (\mathfrak{s}(y) - \mathfrak{s}(q_\circ))\mathfrak{m}(y)\, dy + \frac{\mathfrak{s}(q) - \mathfrak{s}(q_\circ)}{\mathfrak{s}(c) - \mathfrak{s}(q_\circ)} \int_q^c (\mathfrak{s}(c) - \mathfrak{s}(y))\mathfrak{m}(y)\, dy.$$

Send the right barrier $c \uparrow \bar{q}$. Since $\mathfrak{s}(c) \to +\infty$, the first prefactor tends to one, and $\mathfrak{h}_{q_\circ} \wedge \mathfrak{h}_c$ increases to $\mathfrak{h}_{q_\circ}$. In the second term, integration by parts gives $\int_q^c (\mathfrak{s}(c) - \mathfrak{s}(y))\mathfrak{m}(y)\, dy = \int_q^c \mathfrak{s}'(z)\mathfrak{M}(q, z)\, dz$, with $\mathfrak{M}(q, z) = \int_q^z \mathfrak{m}$. Dividing by $\mathfrak{s}(c) - \mathfrak{s}(q_\circ) = \int_{q_\circ}^c \mathfrak{s}'$ gives an $\mathfrak{s}'$-weighted average over $(q_\circ, c)$ of the bounded function that equals $\mathfrak{M}(q, z)$ for $z > q$ and zero otherwise. Because $\mathfrak{s}'$ is not integrable at $\bar{q}$, the weight concentrates near c, and finite upper speed mass gives the limit $\int_q^{\bar{q}} \mathfrak{m}$. This mirror image of the argument at the origin yields

$$\mathbb{E}_q[\mathfrak{h}_{q_\circ}] = (\mathfrak{s}(q) - \mathfrak{s}(q_\circ)) \int_q^{\bar{q}} \mathfrak{m}(y)\, dy + \int_{q_\circ}^{q} (\mathfrak{s}(y) - \mathfrak{s}(q_\circ))\mathfrak{m}(y)\, dy. \tag{A.37}$$

The first piece is remaining scale from $q_\circ$ to q times the speed tail from q to $\bar{q}$. With

$$\mathfrak{s}(q) - \mathfrak{s}(q_\circ) = \int_{q_\circ}^{q} \mathfrak{s}'(\tilde{q})\, d\tilde{q},$$

fix $q_1 \in (q_\circ, \bar{q})$ as above. The integral from $q_\circ$ to q_1 is a finite constant. On (q_1, q) use $\mathfrak{s}'(\tilde{q}) \sim C_{\mathfrak{s},u}(\bar{q} - \tilde{q})^{\lambda_u} \exp[C_u/(\bar{q} - \tilde{q})]$ and $w = \bar{q} - \tilde{q}$ to obtain

$$\int_{q_1}^{q} \mathfrak{s}'(\tilde{q})\, d\tilde{q} \sim C_{\mathfrak{s},u} \int_{\bar{q}-q}^{\bar{q}-q_1} w^{\lambda_u} \exp\left[\frac{C_u}{w}\right] dw.$$

The integrand explodes as $w \downarrow 0$. Set $\Xi(w) = w^{\lambda_u + 2} \exp(C_u/w)$. Then

$$\Xi'(w) = w^{\lambda_u} \exp\left[\frac{C_u}{w}\right] ((\lambda_u + 2)w - C_u),$$

hence $w^{\lambda_u}\exp(C_u/w) = \Xi'(w)/((\lambda_u+2)w - C_u)$. For small w the denominator is $-C_u + O(w)$, and the definite integral from $\bar{w} = \bar{q} - q$ to the fixed $\bar{q} - q_1$ is therefore

$$\int_{\bar{w}}^{\bar{q}-q_1} w^{\lambda_u} e^{C_u/w}\, dw = \frac{\Xi(\bar{w})}{C_u}(1 + O(\bar{w})) = \frac{\bar{w}^{\lambda_u+2}}{C_u}\exp\left[\frac{C_u}{\bar{w}}\right](1 + O(\bar{w})),$$

up to a $\bar{w}$-independent constant from the upper limit. Remaining scale is therefore

$$\mathfrak{s}(q) - \mathfrak{s}(q_\circ) \sim \frac{C_{\mathfrak{s},u}}{C_u}(\bar{q} - q)^{\lambda_u+2}\exp\left[\frac{C_u}{\bar{q} - q}\right].$$

The speed tail already computed in (A.36) is

$$\int_q^{\bar{q}} \mathfrak{m}(y)\, dy \sim C_{u,\star}(\bar{q} - q)^{-\lambda_u}\exp\left[-\frac{C_u}{\bar{q} - q}\right].$$

The product of these two equivalents is

$$(\mathfrak{s}(q) - \mathfrak{s}(q_\circ))\int_q^{\bar{q}} \mathfrak{m} \sim \frac{C_{\mathfrak{s},u}C_{u,\star}}{C_u}(\bar{q} - q)^2 \longrightarrow 0.$$

The constant piece of remaining scale, times the same tail, is $O((\bar{q} - q)^{-\lambda_u}\exp[-C_u/(\bar{q} - q)])$ and also tends to 0. Thus the first piece of $\mathbb{E}_q[\mathfrak{h}_{q_\circ}]$ tends to 0.

The second piece is $\int_{q_\circ}^{q}(\mathfrak{s}(y) - \mathfrak{s}(q_\circ))\mathfrak{m}(y)\, dy$. The integrand is nonnegative. Sending $q \uparrow \bar{q}$ enlarges the domain monotonically, so by monotone convergence

$$\lim_{q\uparrow\bar{q}}\int_{q_\circ}^{q}(\mathfrak{s}(y) - \mathfrak{s}(q_\circ))\mathfrak{m}(y)\, dy = \int_{q_\circ}^{\bar{q}}(\mathfrak{s}(y) - \mathfrak{s}(q_\circ))\mathfrak{m}(y)\, dy. \tag{A.38}$$

Write $\mathfrak{s}(y) - \mathfrak{s}(q_\circ) = \int_{q_\circ}^{y}\mathfrak{s}'(z)\, dz$ and apply integration by parts:

$$\int_{q_\circ}^{\bar{q}}\mathfrak{m}(y)\left(\int_{q_\circ}^{y}\mathfrak{s}'(z)\, dz\right)dy = \int_{q_\circ}^{\bar{q}}\mathfrak{s}'(z)\left(\int_z^{\bar{q}}\mathfrak{m}(y)\, dy\right)dz.$$

The inner tail $\int_z^{\bar{q}}\mathfrak{m}(y)\, dy$ is asymptotic to $C_{u,\star}(\bar{q} - z)^{-\lambda_u}\exp[-C_u/(\bar{q} - z)]$. Multiplication by the scale equivalent $\mathfrak{s}'(z) \sim C_{\mathfrak{s},u}(\bar{q} - z)^{\lambda_u}\exp[+C_u/(\bar{q} - z)]$ cancels the two exponentials and the two powers:

$$\mathfrak{s}'(z)\int_z^{\bar{q}}\mathfrak{m} \longrightarrow C_{\mathfrak{s},u}C_{u,\star},$$

which is bounded and integrable on a neighborhood of $\bar{q}$. Hence the integral (A.38) converges. Combined with the vanishing first piece, $\lim_{q\uparrow\bar{q}}\mathbb{E}_q[\mathfrak{h}_{q_\circ}]$ is finite: a path that starts arbitrarily close to $\bar{q}$ reaches $q_\circ$ in finite expected time. Together with inaccessibility from the interior and finite speed mass, $\bar{q}$ is an entrance boundary (Helland, 1996, Theorem 8). The classification permits entrance initialization in a boundary extension; it does not mean that a path started in the interior ever hits $\bar{q}$. Neither endpoint is accessible from the interior.

The invariant density. The same two scale integrals also fix the invariant density. Section A.9 shows that every nonnegative stationary density of this diffusion has the form $\varphi = \mathfrak{m}(C_{\text{stat}} - \mathfrak{j}\,\mathfrak{s})$. The constant $\mathfrak{j}$ is a probability current: the rate at which probability flows past a point. With $\mathfrak{s}(0+) = -\infty$ and $\mathfrak{s}(\bar{q}-) = +\infty$, the only viable current is $\mathfrak{j} = 0$, so $\varphi = C_{\text{stat}}\mathfrak{m}$ with $C_{\text{stat}} > 0$. Since $\mathfrak{m} = 2e^{\mathcal{R}}/\upsilon^2$ and both speed tails are finite, normalization gives (A.14). The existence of an invariant probability, and the fact that it has a density solving the stationary forward equation, are established at the end of the proof rather than taken from the forward equation alone.

Lipschitz coefficients and a strong solution. The endpoint integrals show that neither boundary is hit from the interior. A strong solution on the whole state interval $(0, \bar{q})$ still requires the coefficients μ and υ to remain Lipschitz as those boundaries are approached, so that the process need not be altered or truncated.

The only possible obstruction is $q \downarrow 0$. Section A.10 shows that the extra factor of q in μ and υ restores bounded derivatives at the origin, and therefore that μ and υ extend Lipschitz-continuously to $[0, \bar{q}]$. Local strong existence and pathwise uniqueness therefore run up to exit, even if the exit time is infinite.

Recurrence, Lyapunov control and ergodicity. The endpoint integrals classify the two boundaries and identify the candidate invariant density.[12] To finish the proof, we show that the process has a drift that pulls the state back from both tails, and that the transition laws overlap, so that the distribution forgets its starting point. Together, these give a unique non-explosive strong solution for all time and convergence in total variation from every interior start. The overlap relies on Section A.11.

A Lyapunov function $\mathcal{V}(q)$ is a scalar test function of the state that becomes large at both endpoints and whose generator is negative when $\mathcal{V}$ is large. If such a function exists, $\mathcal{V}(q_t)$ tends to fall when the path is near 0 or $\bar{q}$.[13] The process is pulled back from the tails on average, which is an input for global non-explosion.

Consider the candidate

$$\mathcal{V}(q) = q^{-\delta_V} + (\bar{q} - q)^{-k_V}. \tag{A.39}$$

Both summands are continuous and strictly positive on $(0, \bar{q})$. As $q \downarrow 0$, $q^{-\delta_V} \to +\infty$. As $q \uparrow \bar{q}$, $(\bar{q} - q)^{-k_V} \to +\infty$. A continuous function on an open interval that diverges at both endpoints attains a global minimum at an interior point. Pick any $q_\star \in (0, \bar{q})$ and choose $\epsilon > 0$ small enough that $\mathcal{V} > \mathcal{V}(q_\star)$ on $(0, \epsilon] \cup [\bar{q} - \epsilon, \bar{q})$. The restriction of $\mathcal{V}$ to the compact interval $[\epsilon, \bar{q} - \epsilon]$ attains its minimum, and that value is the global minimum on $(0, \bar{q})$. Both summands are positive,

[12] The invariant density is identified once diverging scale rules out a nonzero probability current.
[13] This is a Foster–Lyapunov drift condition (Meyn and Tweedie, 1993).

so the minimum is positive, and $\mathcal{V}$ is bounded below by a positive constant. Its sublevel sets $\{\mathcal{V} \leq n\}$ are closed subsets of $[\epsilon_n, \bar{q} - \epsilon_n]$ for some $\epsilon_n > 0$, hence compact in $(0, \bar{q})$. We impose $0 < \delta_V < \chi_0$, with χ_0 defined in (A.18). This restriction makes the lower-end generator ratio strictly negative. Any $k_V > 0$ works at the upper end, because the inward drift sends the ratio to $-\infty$, as shown below.

We now calculate the generator of each component using $\mathcal{L}f = \mu f' + \upsilon^2 f''/2$. Start with $f(q) = q^{-\delta_V}$:

$$\frac{\mathcal{L}q^{-\delta_V}}{q^{-\delta_V}} = -\delta_V \frac{\mu(q)}{q} + \frac{\delta_V(\delta_V + 1)}{2}(\sigma^s(q))^2.$$

By definition $\ell(q) = \mu(q)/q - (\sigma^s(q))^2/2$,[14] hence $\mu(q)/q = \ell(q) + (\sigma^s(q))^2/2$. Insert this:

$$\frac{\mathcal{L}q^{-\delta_V}}{q^{-\delta_V}} = -\delta_V \ell(q) + \frac{\delta_V^2}{2}(\sigma^s(q))^2. \tag{A.40}$$

For the second function $f(q) = (\bar{q} - q)^{-k_V}$,

$$\frac{\mathcal{L}(\bar{q} - q)^{-k_V}}{(\bar{q} - q)^{-k_V}} = \frac{k_V \mu(q)}{\bar{q} - q} + \frac{k_V(k_V + 1)\upsilon(q)^2}{2(\bar{q} - q)^2}. \tag{A.41}$$

The generator is linear, so $\mathcal{L}\mathcal{V} = \mathcal{L}q^{-\delta_V} + \mathcal{L}(\bar{q} - q)^{-k_V}$. Write $\mathcal{V}_- = q^{-\delta_V}$ and $\mathcal{V}_+ = (\bar{q} - q)^{-k_V}$. Then

$$\frac{\mathcal{L}\mathcal{V}}{\mathcal{V}} = \frac{\mathcal{L}\mathcal{V}_-}{\mathcal{V}_-} \cdot \frac{\mathcal{V}_-}{\mathcal{V}_- + \mathcal{V}_+} + \frac{\mathcal{L}\mathcal{V}_+}{\mathcal{V}_- + \mathcal{V}_+}.$$

Near zero, $\mathcal{V}_+ \to \bar{q}^{-k_V}$ and $\mathcal{L}\mathcal{V}_+$ stays bounded, because $(\bar{q} - q)^{-k_V}$ is C^2 on a neighborhood of 0 inside $(0, \bar{q})$ and μ and υ are bounded there. Also $\mathcal{V}_- \to +\infty$, so $\mathcal{V}_-/(\mathcal{V}_- + \mathcal{V}_+) \to 1$ and $(\mathcal{L}\mathcal{V}_+)/(\mathcal{V}_- + \mathcal{V}_+) \to 0$. Hence

$$\frac{\mathcal{L}\mathcal{V}(q)}{\mathcal{V}(q)} \longrightarrow -\delta_V \ell_0 + \frac{\delta_V^2}{2}(\bar{\sigma}^s)^2,$$

where $\ell(q) \to \ell_0$ from (A.17). Rewrite the limit with $\chi_0 = 2\ell_0/(\bar{\sigma}^s)^2$:

$$-\delta_V \ell_0 + \frac{\delta_V^2}{2}(\bar{\sigma}^s)^2 = \frac{\delta_V(\bar{\sigma}^s)^2}{2}\left(\delta_V - \frac{2\ell_0}{(\bar{\sigma}^s)^2}\right) = \frac{\delta_V(\bar{\sigma}^s)^2}{2}(\delta_V - \chi_0).$$

The restriction $0 < \delta_V < \chi_0$ makes this limit strictly negative.

Near $\bar{q}$ the roles reverse. $\mathcal{V}_-$ and $\mathcal{L}\mathcal{V}_-$ stay bounded, while $\mathcal{V}_+ \to +\infty$. The limits (A.32) and (A.33), $\mu \to \mu_u < 0$ and $\upsilon \sim \bar{q} c_u (\bar{q} - q)$, give

$$\frac{\mathcal{L}\mathcal{V}(q)}{\mathcal{V}(q)} = \frac{k_V \mu_u}{\bar{q} - q} + O(1).$$

[14] Recall that ℓ is the drift of $\log q$, defined in (A.15).

The first term tends to $-\infty$ and dominates the bounded remainder, whose limit is $k_V(k_V+1)\bar{q}^2c_u^2/2$. Thus the ratio for the full $\mathcal{V}$ tends to $-\infty$ at $\bar{q}$.

Choose neighborhoods of 0 and of $\bar{q}$ on which the ratio is at most $-c_{\mathcal{V}}$ for some $c_{\mathcal{V}} > 0$. Then $\mathcal{L}\mathcal{V} \leq -c_{\mathcal{V}}\mathcal{V}$ there. On the remaining compact subinterval both $\mathcal{V}$ and $\mathcal{L}\mathcal{V}$ are continuous, so $\mathcal{L}\mathcal{V} + c_{\mathcal{V}}\mathcal{V}$ is bounded above by a finite constant $K_{\mathcal{V}} > 0$. Combining the two regions yields the Foster inequality, which has the form of condition (CD3) in Meyn and Tweedie (1993) and implies their non-explosion condition of Theorem 2.1,

$$\mathcal{L}\mathcal{V}(q) \leq -c_{\mathcal{V}}\mathcal{V}(q) + K_{\mathcal{V}} \qquad \text{for all } q \in (0, \bar{q}). \tag{A.42}$$

The constants $c_{\mathcal{V}}$ and $K_{\mathcal{V}}$ can depend on the primitive parameters. The additive $K_{\mathcal{V}}$ absorbs the compact middle of the interval, where the process need not be pulled. The negative multiple of $\mathcal{V}$ is the pull in the two tails.

The process has a unique strong solution only up to its first exit from $(0, \bar{q})$. Localization turns the Foster inequality into a bound that prevents that exit. Because $\mathcal{V}$ is continuous and proper, the set $\{\mathcal{V} \geq n\}$ lies in a pair of neighborhoods of 0 and $\bar{q}$ and does not meet a fixed interior compact set. For an integer $n > \mathcal{V}(q)$, let $\mathfrak{t}_n$ be the first time $\mathcal{V}(q_t)$ equals n, and fix a horizon $T < \infty$. On $[0, \mathfrak{t}_n]$ the path stays in a compact subset of $(0, \bar{q})$, where μ and υ are Lipschitz and $\mathcal{V}$ is C^2 and bounded. Apply (A.79) to $f = \mathcal{V}$:

$$d\mathcal{V}(q_t) = \mathcal{L}\mathcal{V}(q_t)\, dt + \upsilon(q_t)\mathcal{V}'(q_t)\, dZ_t.$$

Integrate from date 0 to the stopped time $T \wedge \mathfrak{t}_n$:

$$\mathcal{V}(q_{T\wedge\mathfrak{t}_n}) = \mathcal{V}(q) + \int_0^{T\wedge\mathfrak{t}_n} \mathcal{L}\mathcal{V}(q_s)\, ds + \int_0^{T\wedge\mathfrak{t}_n} \upsilon(q_s)\mathcal{V}'(q_s)\, dZ_s.$$

On $[0, \mathfrak{t}_n]$ the state remains in a compact subset of $(0, \bar{q})$, where υ and $\mathcal{V}'$ are continuous, hence bounded. The integrand of the stochastic integral is therefore square-integrable on $[0, T]$, so the stopped integral is a true martingale with mean zero under $\mathbb{E}_q$. Taking expectations leaves only the generator integral:

$$\mathbb{E}_q\big[\mathcal{V}(q_{T\wedge\mathfrak{t}_n})\big] = \mathcal{V}(q) + \mathbb{E}_q\left[\int_0^{T\wedge\mathfrak{t}_n} \mathcal{L}\mathcal{V}(q_s)\, ds\right].$$

The Foster inequality (A.42) implies $\mathcal{L}\mathcal{V}(q) \leq K_{\mathcal{V}}$ at every interior state, since $\mathcal{V} > 0$. The integrand is therefore at most $K_{\mathcal{V}}$, and the length of the integration interval is at most T:

$$\int_0^{T\wedge\mathfrak{t}_n} \mathcal{L}\mathcal{V}(q_s)\, ds \leq K_{\mathcal{V}}(T \wedge \mathfrak{t}_n) \leq K_{\mathcal{V}}T, \qquad \text{so} \qquad \mathbb{E}_q\big[\mathcal{V}(q_{T\wedge\mathfrak{t}_n})\big] \leq \mathcal{V}(q) + K_{\mathcal{V}}T.$$

On the event $\{\mathfrak{t}_n \le T\}$, continuity gives $\mathcal{V}(q_{\mathfrak{t}_n}) = n$. Since $\mathcal{V} \ge 0$,

$$n\,\mathbb{P}_q(\mathfrak{t}_n \le T) \le \mathbb{E}_q\big[\mathcal{V}(q_{T\wedge\mathfrak{t}_n})\big] \le \mathcal{V}(q) + K_{\mathcal{V}}T, \qquad \text{so} \qquad \mathbb{P}_q(\mathfrak{t}_n \le T) \le \frac{\mathcal{V}(q) + K_{\mathcal{V}}T}{n}.$$

As $n \to \infty$, the right side tends to 0 for every finite T. The explosion time $\mathfrak{t}_\infty = \lim_n \mathfrak{t}_n$ is therefore infinite almost surely. Since $\mathcal{V}$ diverges at both endpoints, $\mathfrak{t}_\infty$ is also the first exit from $(0, \bar{q})$, so exit cannot occur in finite time. The local solution therefore exists for all $t \ge 0$, and pathwise uniqueness, already obtained up to exit, extends to every finite horizon, without clipping or reflection.[15]

We also obtain the expected Lyapunov bound. Apply (A.79) to the product $e^{c_{\mathcal{V}}t}\mathcal{V}(q_t)$. The product rule and the generator give the drift

$$e^{c_{\mathcal{V}}t}(c_{\mathcal{V}}\mathcal{V}(q_t) + \mathcal{L}\mathcal{V}(q_t)).$$

The Foster inequality (A.42) is exactly $c_{\mathcal{V}}\mathcal{V} + \mathcal{L}\mathcal{V} \le K_{\mathcal{V}}$. Stopping at $t \wedge \mathfrak{t}_n$ again makes the stochastic integral a martingale, so

$$\mathbb{E}_q\big[e^{c_{\mathcal{V}}(t\wedge\mathfrak{t}_n)}\mathcal{V}(q_{t\wedge\mathfrak{t}_n})\big] \le \mathcal{V}(q) + K_{\mathcal{V}}\mathbb{E}_q\left[\int_0^{t\wedge\mathfrak{t}_n} e^{c_{\mathcal{V}}s}\,ds\right] \le \mathcal{V}(q) + \frac{K_{\mathcal{V}}}{c_{\mathcal{V}}}(e^{c_{\mathcal{V}}t} - 1).$$

Send $n \to \infty$. Almost sure non-explosion gives $t \wedge \mathfrak{t}_n \to t$ and $q_{t\wedge\mathfrak{t}_n} \to q_t$. Fatou's lemma passes the expectation to the left side and yields $\mathbb{E}_q[e^{c_{\mathcal{V}}t}\mathcal{V}(q_t)] \le \mathcal{V}(q) + (K_{\mathcal{V}}/c_{\mathcal{V}})(e^{c_{\mathcal{V}}t} - 1)$. Multiply by $e^{-c_{\mathcal{V}}t}$:

$$\mathbb{E}_q\mathcal{V}(q_t) \le e^{-c_{\mathcal{V}}t}\mathcal{V}(q) + \frac{K_{\mathcal{V}}}{c_{\mathcal{V}}}(1 - e^{-c_{\mathcal{V}}t}). \tag{A.43}$$

The expected Lyapunov value therefore stays finite at every finite time and is bounded uniformly in t by $\mathcal{V}(q) + K_{\mathcal{V}}/c_{\mathcal{V}}$. This argument proves global existence separately from any assertion about an invariant distribution.

We now turn to convergence of the distribution of q. Lyapunov control pulls the process back from the tails on average, but that pullback alone does not give nearby interior starts overlapping successor laws, so the distribution might not forget its initial condition. Harris's theorem (Hairer and Mattingly, 2011; Hairer, 2018) supplies the missing ingredient: besides the Foster bound, it requires that overlap.

Fix a sampling interval $\Delta > 0$ and write $P_t(q, \cdot)$ for the transition law. A set C is small, in the sense used below, if the one-step kernel from every point of C is bounded below by a common

[15] This result is consistent with the scale classification above, where infinite scale distance already makes both endpoints inaccessible from the interior.

positive multiple of a single probability measure ν_C:[16]

$$P_\Delta(q, A) \geq \epsilon_C \nu_C(A) \qquad \text{for every } q \in C \text{ and every Borel } A. \tag{A.44}$$

The interpretation is straightforward. After one sampling step, two trajectories that both sit in C can be coupled onto the ν_C-component with probability ϵ_C, and from that instant they can be driven by the same process, so that each return to C is a chance to forget the exact interior location. Together with the Foster inequality, which makes such returns occur with uniformly integrable waiting times, this erases the memory of the initial state. That guarantees uniqueness of the invariant probability and convergence in total variation. Without the overlap, Lyapunov pullback alone could still permit two interior basins that never mix.

The candidate small sets are Lyapunov sublevels. The expected Lyapunov bound (A.43) at time Δ gives the sampled drift

$$P_\Delta \mathcal{V}(q) \leq e^{-c_\mathcal{V}\Delta}\mathcal{V}(q) + K_\Delta, \qquad K_\Delta = \frac{K_\mathcal{V}}{c_\mathcal{V}}(1 - e^{-c_\mathcal{V}\Delta}). \tag{A.45}$$

The contraction factor $e^{-c_\mathcal{V}\Delta}$ is strictly smaller than one, and the additive constant is finite. Choose a level $R_\mathcal{V} > \max\{\min \mathcal{V}, 2K_\Delta/(1 - e^{-c_\mathcal{V}\Delta})\}$ and let $C = \{q : \mathcal{V}(q) \leq R_\mathcal{V}\}$. Properness makes C compact in $(0, \bar{q})$, and strict convexity of $\mathcal{V}$ makes it a compact interval with positive length. Enclose C in a strictly larger compact interval $I \subset (0, \bar{q})$, with C in the interior of I. On I, the loading υ is continuous and bounded away from zero, because $\sigma^s > 0$ on $(0, \bar{q})$ and q stays away from both endpoints; positivity and compactness, not any lower bound of σ^s by σ, supply this nondegeneracy. The larger interval I is used only as room in which paths may wiggle.

A path is killed at its first exit from I, or from its image I_x below: from that time onward it is discarded and its position is no longer recorded. The killed transition density at time Δ is the subprobability density of being at an interior point without having left the interval: it equals the ordinary density restricted to paths that stayed inside throughout $[0, \Delta]$. Its integral over the interval is the probability of no exit, which is strictly less than one.[17]

Section A.11 passes to the Lamperti coordinate $x(q) = \int_{q_\circ}^{q} \upsilon^{-1}$ on the interval I just fixed, removes the state dependence of the diffusion coefficient, and identifies the killed physical path measure with the killed Brownian measure reweighted by the stopped exponential martingale $\mathcal{E}_\Delta$. It produces a constant $\epsilon_\mathcal{E} > 0$, depending only on Δ, C and I, such that $\mathcal{E}_\Delta \geq \epsilon_\mathcal{E}$ on surviving paths with endpoints in $x(C)$. The rest of the argument stays on the q-axis and turns that bound into the constant ϵ_C in (A.44).

Write $p^\circ(\Delta, x, y)$ for the killed Brownian transition density on I_x: the density of occupying

[16] Inequality (A.44) does not imply that every start in C has the same successor law. It says that those laws share a common lump of mass ϵ_C, distributed exactly as ν_C, with leftover mass $1 - \epsilon_C$ that may still depend on the start.

[17] The true process is not killed. It may leave I and return.

y at date Δ without having left I_x on $[0,\Delta]$. It factors as the free Gaussian density times the probability that the Brownian bridge from x to y in time Δ stays inside I_x:

$$p^\circ(\Delta,x,y) = (2\pi\Delta)^{-1/2}\exp(-(y-x)^2/(2\Delta))\cdot\mathbb{P}(\text{bridge from } x \text{ to } y \text{ stays in } I_x). \tag{A.46}$$

On the compact product $x(C)\times x(C)$, the Gaussian is continuous and positive. The distance $|y-x|$ cannot exceed the diameter of $x(C)$, so the Gaussian is at least

$$\epsilon_{\mathrm{G}} := (2\pi\Delta)^{-1/2}\exp(-\mathrm{diam}(x(C))^2/(2\Delta)) > 0.$$

The second factor in (A.46) is the probability that the path stays in I_x, conditional on the endpoints (x,y). That conditional process is the Brownian bridge from x to y in time Δ: free Brownian motion on $[0,\Delta]$, restricted to the event $\{W_\Delta = y\}$ when $W_0 = x$. The standard decomposition separates a deterministic segment from a mean-zero fluctuation. Let W be a free Brownian motion starting at 0, and set $\mathring{W}_s = W_s - (s/\Delta)W_\Delta$. Then $\mathring{W}_0 = \mathring{W}_\Delta = 0$, and the bridge is

$$X_s^{x\to y} = x + \frac{s}{\Delta}(y-x) + \mathring{W}_s, \qquad s\in[0,\Delta].$$

The first two terms trace the straight segment from x to y. The process $\mathring{W}$ is a standard Brownian bridge from 0 to 0, and its law does not depend on the endpoints (x,y).

Write the compact interval I_x as $[x_L, x_R]$. The set $x(C)$, the image of C under the Lamperti coordinate, lies in a strictly smaller compact subinterval, because C was enclosed in the interior of I. Hence there is a gap $d>0$ such that every point of $x(C)$ lies in $[x_L+d, x_R-d]$. The segment joining any $x,y\in x(C)$ is the ordinary interval between them, so it also remains in $[x_L+d,x_R-d]$. If $\sup_{s\le\Delta}|\mathring{W}_s| < d$, then

$$x_L < x + \frac{s}{\Delta}(y-x) + \mathring{W}_s < x_R \qquad \text{for every } s\in[0,\Delta],$$

so the whole bridge stays in I_x. Therefore

$$\mathbb{P}(X^{x\to y} \text{ stays in } I_x) \ge \mathbb{P}(\sup_{s\le\Delta}|\mathring{W}_s| < d).$$

The right side does not depend on (x,y). It is positive because the Brownian bridge pinned at 0 at both ends remains in $(-d,d)$ on a compact time interval with positive probability. Call that probability $\epsilon_{\mathrm{br}} > 0$. The same d and the same $\mathring{W}$ work for every pair in $x(C)\times x(C)$, so the lower bound is uniform. Hence

$$p^\circ(\Delta,x,y) \ge \epsilon_{\mathrm{G}}\epsilon_{\mathrm{br}} =: \epsilon^\circ > 0 \qquad \text{for all } x,y\in x(C). \tag{A.47}$$

Section A.11 changes Brownian motion into the Lamperti process by Girsanov's theorem:

each surviving path is weighted by the stopped exponential martingale $\mathcal{E}_\Delta$. The lower bound (A.47) is for the killed Brownian density $p^\circ(\Delta, x, y)$ of $dx = dW$ until exit. We convert p° into the killed Lamperti density $p^{\text{kill}}(\Delta, x, y)$ of $dx = \mathfrak{b}(x)\,dt + dW$ until exit, which is the density of the physical x-process, and bound it below. On surviving paths, $d\mathrm{Q}_{\text{phys}} = \mathcal{E}_\Delta\, d\mathrm{P}$. Therefore the subprobability of occupying y at date Δ without having left I_x is

$$p^{\text{kill}}(\Delta, x, y)\,dy = \mathbb{E}_{\mathrm{P}}\big[1_{\{x_\Delta \in dy,\,\text{no exit}\}}\mathcal{E}_\Delta\big] = \mathbb{E}_{\mathrm{P}}\big[\mathcal{E}_\Delta \mid x_\Delta = y,\,\text{no exit}\big]\,p^\circ(\Delta, x, y)\,dy.$$

The conditional expectation is an average of $\mathcal{E}_\Delta$ over Brownian bridges from x to y that stay in I_x. On every such path $\mathcal{E}_\Delta \geq \epsilon_{\mathcal{E}}$ (Section A.11), so the average is at least $\epsilon_{\mathcal{E}}$. Hence

$$p^{\text{kill}}(\Delta, x, y) = \mathbb{E}_{\mathrm{P}}\big[\mathcal{E}_\Delta \mid x_\Delta = y,\,\text{no exit}\big]\,p^\circ(\Delta, x, y) \geq \epsilon_{\mathcal{E}}\,p^\circ(\Delta, x, y) \geq \epsilon_{\mathcal{E}}\epsilon^\circ \tag{A.48}$$

uniformly for $x, y \in x(C)$. Since the true Lamperti process is not killed, any path that leaves I_x and returns is counted by the true transition law and ignored by the killed density, so the true law dominates the killed one.

Transforming the density from x to q multiplies by the Jacobian $x'(q) = 1/\upsilon(q)$. If p_x is a density in the x-coordinate, the corresponding q-density is $p_q(q_0, q) = p_x(x(q_0), x(q))/\upsilon(q)$. On C the loading υ is continuous and positive, hence bounded above by some $\upsilon_C < \infty$, and $1/\upsilon(q) \geq 1/\upsilon_C > 0$. Therefore, for every $q_0, q \in C$, the killed q-transition density obeys

$$\frac{p^{\text{kill}}(\Delta, x(q_0), x(q))}{\upsilon(q)} \geq \frac{\epsilon_{\mathcal{E}}\epsilon^\circ}{\upsilon_C} =: c_C > 0. \tag{A.49}$$

The lower bound c_C is the product of the Girsanov lower bound $\epsilon_{\mathcal{E}}$, the killed Brownian lower bound ϵ° and the reciprocal of the maximal loading on C. It depends on Δ, C and I, but not on the starting point inside C. Integrating (A.49) over $A \cap C$ and using that the true law dominates the killed one gives, for every $q_0 \in C$ and Borel A,

$$P_\Delta(q_0, A) \geq c_C\,\mathrm{Leb}(A \cap C) \geq \epsilon_C \nu_C(A), \quad \nu_C(A) = \frac{\mathrm{Leb}(A \cap C)}{\mathrm{Leb}(C)}, \quad \epsilon_C = \min\left\{\frac{1}{2}, c_C\,\mathrm{Leb}(C)\right\}. \tag{A.50}$$

Here ν_C is a probability measure and $0 < \epsilon_C < 1$, so (A.44) holds.

Existence, identification and convergence of the invariant law. The sampled drift (A.45) is Assumption 1 of Hairer and Mattingly (2011), and the minorization (A.50) on the sublevel set $C = \{\mathcal{V} \leq R_{\mathcal{V}}\}$, with $R_{\mathcal{V}} > 2K_\Delta/(1 - e^{-c_{\mathcal{V}}\Delta})$, is their Assumption 2. Their Theorem 1.2 yields a unique invariant probability for P_Δ, and their Theorem 1.3 gives geometric convergence to it from every interior point in a weighted total-variation norm, hence in total variation. This invariant probability is invariant for every P_t: by the semigroup property, its image under P_t is

also invariant for P_Δ, and uniqueness makes the two equal.

The invariant probability has an interior density. On any compact interior interval, the killed diffusion has a density by the preceding change of measure. Non-exit and continuity let these intervals exhaust every path on a finite horizon, giving absolute continuity of each transition law and hence of any invariant probability. Interior smoothness and nondegeneracy give the stationary forward equation. The scale limits established above force its current to vanish in (A.86). Since $\mathfrak{m} = 2e^{\mathcal{R}}/v^2$ and its total mass is $2\mathcal{N}$, normalization gives exactly (A.14). In particular, the density has full interior support and a finite absolute log moment.

If $t = n\Delta + r$ with $0 \le r < \Delta$, total-variation contraction by a Markov kernel gives

$$\|P_t(q_0,\cdot) - \varphi_\star(q)dq\|_{\mathrm{TV}} \le \|P_{n\Delta}(q_0,\cdot) - \varphi_\star(q)dq\|_{\mathrm{TV}} \longrightarrow 0.$$

Finally, uniqueness makes the invariant probability ergodic (Hairer, 2018); Proposition A.4 uses this to obtain almost-sure convergence of time averages. The nondegeneracy and joint stationarity of the output gap, centered inflation and excess volatility under invariant initialization are shown in Section A.6. This completes the proof of Proposition A.1. □

A.6 Stationarity versus initialization

The graph is strictly increasing, and the excess-volatility function is continuous and nonconstant, positive in the interior and zero at $\bar{q}$. Since the invariant density has full interior support, their invariant distributions are nondegenerate, as is that of $q-1$. Invariant initialization draws q_0 from (A.14) independently of future Brownian increments and includes it in time-zero information. In that case, the IS equation, the Phillips curve and the policy rule hold at every date, and $(q_t - 1, \widehat{\pi}(q_t), \sigma^s(q_t))$ is jointly strictly stationary. The corresponding actual inflation is $\bar{\pi} + \widehat{\pi}(q_t)$. For fixed (b,ζ), Proposition A.1 gives uniqueness of the invariant law. Different pairs (b,ζ) give distinct joint stationary laws: equality would require their centered-inflation graphs to agree on an open interval of full-support states, which forces equality of both their positive amplitudes and their exponents.

The main text instead permits a prescribed interior q_0, corroborated by initial money when the rule selects the stochastic equilibrium. Such a process is not strictly stationary at date zero. It has the invariant law just derived and converges to it.[18]

The log moment also has a finite-horizon counterpart. Boundedness of ℓ and σ^s in (A.15) gives

$$\mathbb{E}_{q_0}|\log q_t| \le |\log q_0| + t\sup|\ell| + \sqrt{t}\sup\sigma^s.$$

Since $\log C_t = \log A_t + \log L^n + \log q_t$ and labor is bounded, expected discounted absolute flow

[18] Note that this fixed q_0 is different from random invariant initialization, which is a separate convention requiring initial randomness in the filtration; it is not supplied by a Brownian filtration with trivial time-zero information.

utility is finite from every fixed interior start.[19]

A.7 Properties of the stochastic family

The functions constructed in Section A.2 determine the range of each observable and whether it rises or falls with the gap. Define the nominal policy-rate function and the real-rate function along the graph,

$$i(q) = r^n + \bar{\pi} + \phi_y(q-1) + \phi_\pi \widehat{\pi}(q), \qquad r(q) = i(q) - \bar{\pi} - \widehat{\pi}(q) = r^n + J(q).$$

The limit of excess volatility at zero is $\bar{\sigma}^s = \lim_{q \downarrow 0} \sigma^s(q)$, given by (A.16). For bounded stationary observables f_1 and f_2, write $\mathrm{Cov}_\star(f_1, f_2) = \mathbb{E}_\star[f_1 f_2] - \mathbb{E}_\star[f_1]\mathbb{E}_\star[f_2]$.

Proposition A.2 (Bounds and cyclical comovement)**.** *At every interior state, the stochastic solution satisfies*

$$-1 < q - 1 < \bar{q} - 1 < 0, \qquad \underline{\widehat{\pi}} < \widehat{\pi} < \widehat{\pi}(\bar{q}) < 0, \qquad \sigma^s > 0. \tag{A.51}$$

Labor, marginal cost and the ratio of profit and rebate income to nominal consumption are bounded, with

$$0 < L < L^n \bar{q}, \qquad 0 < \mathrm{mc} < \alpha \bar{q}^{1+1/\eta}, \qquad 1 - \alpha \bar{q}^{1+1/\eta} < \frac{D}{pC} < 1. \tag{A.52}$$

Centered inflation strictly increases with the output gap, and $\mathrm{Cov}_\star(q-1, \widehat{\pi}) > 0$ *under the invariant law. If, in addition,* $\phi_y \geq 0$ *and* $\phi_\pi \geq 1$*, not both with equality, then*

$$0 < \sigma^s < \bar{\sigma}^s, \qquad -\Phi < r - r^n < J(\bar{q}) < 0, \tag{A.53}$$

both interest rates strictly increase with the output gap, excess volatility strictly decreases, and under the invariant law

$$\mathrm{Cov}_\star(q-1, \sigma^s) < 0, \qquad \mathrm{Cov}_\star(\widehat{\pi}, \sigma^s) < 0. \tag{A.54}$$

Proof. By Lemma A.1, $0 < q < \bar{q} < 1$, $\widehat{\pi}(q) < 0$ and $\sigma^s(q) > 0$ on the state interval. Differentiation gives

$$\begin{aligned} \widehat{\pi}'(q) &= \beta\zeta q^{\zeta-1} > 0, & J'(q) &= \phi_y + (\phi_\pi - 1)\widehat{\pi}'(q), \\ i'(q) &= \phi_y + \phi_\pi \widehat{\pi}'(q), & (\sigma^s)'(q) &= \frac{F'(q)}{\sigma + (1+\zeta)\sigma^s(q)}. \end{aligned} \tag{A.55}$$

The limits at zero and $\bar{q}$, with $\widehat{\pi}(0+) = \underline{\widehat{\pi}}$, prove (A.51). In the limit, $L = L^n q$, and equation

[19] This integrability condition does not replace the individual optimality conditions in Section B.1.

(A.4) with $H_{\gamma,t} = 1$ gives $\text{mc} = L^{1+1/\eta} = \alpha q^{1+1/\eta}$, because $(L^n)^{1+1/\eta} = \alpha$. By (B.24) below, $D/(pC) = 1 - L^{1+1/\eta} = 1 - \alpha q^{1+1/\eta}$. These prove (A.52).

Now let $\phi_y \geq 0$ and $\phi_\pi \geq 1$, not both with equality. Lemma A.1 gives $F' < 0$ and $J < 0$ on $(0, 1)$. Hence $J' > 0$, $i' \geq J' + \widehat{\pi}' > 0$ and $(\sigma^s)' < 0$. The limits $J(0+) = -\Phi$, $\sigma^s(0+) = \bar{\sigma}^s$ and $\sigma^s(\bar{q}) = 0$ then prove (A.53).

The gap $q - 1$ and centered inflation $\widehat{\pi}(q)$ are strictly increasing in q, and in the restricted case excess volatility is strictly decreasing. Each function is bounded on $(0, \bar{q})$. Under a law with strictly positive density, the covariance of two strictly increasing functions of the state is strictly positive, and the covariance of a strictly increasing function and a strictly decreasing function is strictly negative. Full interior support of the invariant density (Proposition A.1) therefore proves the covariance signs. □

Every interior state of every stochastic solution features underproduction and below-target inflation, and centered inflation is procyclical in q. When $\phi_y \geq 0$ and $\phi_\pi \geq 1$, every interior state also features a real rate below its natural counterpart, $J(q) = r(q) - r^n < 0$, and excess volatility is countercyclical. Without these signs, the real rate can exceed its natural counterpart near zero output (Lemma A.4). These comovements compare states under the invariant distribution at a given date.

Lemma A.4 (Real rate above the natural rate)**.** *In every stochastic solution, the real rate lies below the natural rate near $\bar{q}$, and it exceeds the natural rate near zero output if $\Phi < 0$. For every $\phi_y \in (-\rho, 0)$, there are values of ϕ_π for which the rule satisfies both conditions of (3.5), $\Phi < 0$, and shape pairs exist.*

Proof. J is continuous on $(0, \bar{q}]$. The endpoint condition $J(\bar{q}) < 0$ in (A.6) gives $r(q) < r^n$ near $\bar{q}$, and $J(0+) = -\Phi$ gives $r(q) > r^n$ near $q \simeq 0$ when $\Phi < 0$. Since $\omega^2 - \rho^2 = 2(\varepsilon - 1)\kappa$ and $\underline{\widehat{\pi}} = -(\omega - \rho)/(\varepsilon - 1)$,

$$-\frac{\rho \underline{\widehat{\pi}}}{(1 + 1/\eta)\kappa} = \frac{2\rho}{(1 + 1/\eta)(\omega + \rho)} < 1,$$

because $\omega > \rho$. Fix $\phi_y \in (-\rho, 0)$. Dividing this inequality by $-\underline{\widehat{\pi}} > 0$ and multiplying by $-\phi_y > 0$ shows that the interval

$$1 - \frac{\rho \phi_y}{(1 + 1/\eta)\kappa} < \phi_\pi < 1 + \frac{\phi_y}{\underline{\widehat{\pi}}}$$

is nonempty. On it, the first inequality is the Taylor principle, the trace condition holds because $\phi_y > -\rho$, and the second inequality is equivalent to $\Phi = \phi_y - (\phi_\pi - 1)\underline{\widehat{\pi}} < 0$. Lemma A.2 then supplies shape pairs. □

A.7.1 Frisch elasticity of labor supply

The curvature of labor disutility, governed by the Frisch elasticity η, moves the state interval $(0, \bar{q})$ without moving the centered-inflation graph.

Corollary A.1 (Labor elasticity)**.** *Hold the policy coefficients and the shape pair (b, ζ) fixed, and let $\eta_1 < \eta_2$ be two Frisch elasticities at which (b, ζ) satisfies the endpoint conditions (A.6). Write $\bar{q}_\eta$ and σ^s_η for the corresponding upper endpoint and excess volatility. Then $\bar{q}_{\eta_1} > \bar{q}_{\eta_2}$ and $\sigma^s_{\eta_1}(q) > \sigma^s_{\eta_2}(q)$ at every $q \in (0, \bar{q}_{\eta_2})$. The centered-inflation function $\widehat{\pi}(q)$ is the same at both elasticities. When $\phi_y \geq 0$ and $\phi_\pi \geq 1$, not both with equality, the endpoint conditions hold at every η.*

Proof. The constants (A.1) and the shape conditions (A.2) do not depend on η, because $\kappa = (\varepsilon - 1)/\tau$ does not. If $\eta_1 < \eta_2$, then $1 + 1/\eta_1 > 1 + 1/\eta_2$, so $q^{1+1/\eta_1-\zeta} < q^{1+1/\eta_2-\zeta}$ for $0 < q < 1$. Equation (A.4) gives $F_{\eta_1}(q) > F_{\eta_2}(q)$ on $(0, 1)$. Since $F_{\eta_2} \geq 0$ on $(0, \bar{q}_{\eta_2}]$, $F_{\eta_1} > 0$ there, so the smallest zero of F_{η_1} exceeds $\bar{q}_{\eta_2}$. The volatility root in equation (3.6) is strictly increasing in F, which proves the volatility comparison. The graph $\widehat{\pi}(q) = \underline{\widehat{\pi}} + \beta q^\zeta$ does not depend on η. The last claim follows from Lemma A.1. □

Thus, within this family, the Frisch elasticity shifts the invariant distribution and its domain,[20] without moving the centered-inflation graph itself.

A.7.2 Expected gap and inflation

For any nonnegative real exponent ϖ, define the invariant moment from (A.14):

$$M_\varpi \equiv \mathbb{E}_\star[q^\varpi] = \frac{\int_0^{\bar{q}} q^\varpi e^{\mathcal{R}(q)} \upsilon(q)^{-2} dq}{\int_0^{\bar{q}} e^{\mathcal{R}(q)} \upsilon(q)^{-2} dq}. \tag{A.56}$$

Proposition A.1 guarantees that M_ϖ is well defined.

Proposition A.3 (Stationary means and drift restrictions)**.** *The stationary expected proportional gap and centered inflation are*

$$\mathbb{E}_\star[q - 1] = M_1 - 1, \qquad \mathbb{E}_\star[\widehat{\pi}] = \underline{\widehat{\pi}} + \beta M_\zeta. \tag{A.57}$$

They are strictly negative and strictly between the corresponding bounds in (A.51). They satisfy

[20] It also changes natural labor $L^n = \alpha^{\eta/(1+\eta)}$.

the exact restrictions

$$\phi_y \mathbb{E}_\star[q-1] + (\phi_\pi - 1)\mathbb{E}_\star[\widehat{\pi}] + \sigma \mathbb{E}_\star[\sigma^s] + \frac{1}{2}\mathbb{E}_\star[(\sigma^s)^2] = 0, \tag{A.58}$$

$$\rho \mathbb{E}_\star[\widehat{\pi}] - \kappa(M_{1+1/\eta} - 1) - \frac{\varepsilon - 1}{2}\mathbb{E}_\star[\widehat{\pi}^2] = 0. \tag{A.59}$$

In particular,

$$\mathbb{E}_\star[\widehat{\pi}] = \frac{\kappa}{\rho}(M_{1+1/\eta} - 1) + \frac{\varepsilon - 1}{2\rho}\mathbb{E}_\star[\widehat{\pi}^2], \tag{A.60}$$

$$\mathbb{E}_\star[r^n - r] = \sigma \mathbb{E}_\star[\sigma^s] + \frac{1}{2}\mathbb{E}_\star[(\sigma^s)^2] > 0, \tag{A.61}$$

and, if $\phi_y \neq 0$,

$$\mathbb{E}_\star[q-1] = -\frac{(\phi_\pi - 1)\mathbb{E}_\star[\widehat{\pi}] + \sigma \mathbb{E}_\star[\sigma^s] + \frac{1}{2}\mathbb{E}_\star[(\sigma^s)^2]}{\phi_y}. \tag{A.62}$$

Nondegeneracy of the invariant density (A.14) *also gives*

$$\mathbb{E}_\star[\widehat{\pi}] < \underline{\widehat{\pi}} + \beta(1 + \mathbb{E}_\star[q-1])^\zeta. \tag{A.63}$$

Proof. Proposition A.1 proves $0 < \mathcal{N} < \infty$. For every $\varpi \geq 0$, q^ϖ is bounded on the state interval, so (A.56) is finite and equals $\mathbb{E}_\star[q^\varpi]$. Integration of $q - 1$ and of the graph gives (A.57). Full interior support and the strict pointwise bounds imply strictly interior means.

The remaining identities concern the invariant law. Realize it on an auxiliary probability space on which q_0 has density $\varphi_\star$ and is independent of the Brownian motion; this device computes integrals against $\varphi_\star$ and requires no change in the main paper's filtration (Section A.6). Then q_T has the same law for every T. Hence

$$\mathbb{E}_\star[\log q_T] = \mathbb{E}_\star[\log q_0],$$

both finite by Proposition A.1. The drift of the log ratio is the bounded function $\ell(q) = J(q) + \sigma\sigma^s(q) + \frac{1}{2}(\sigma^s(q))^2$, and its loading $\sigma^s(q)$ is bounded. Thus

$$\log q_T - \log q_0 = \int_0^T \ell(q_t)\, dt + \int_0^T \sigma^s(q_t)\, dZ_t. \tag{A.64}$$

The stochastic integral is square-integrable, with second moment at most $T \sup(\sigma^s)^2$, and therefore has zero expectation. Taking expectations in (A.64) yields

$$\mathbb{E}_\star[\log q_T] - \mathbb{E}_\star[\log q_0] = \mathbb{E}_\star\left[\int_0^T \ell(q_t)\, dt\right].$$

The left side vanishes by invariance of the law of q_t. Since the integrand is bounded, Fubini's theorem and stationarity imply $0 = T\mathbb{E}_\star[\ell(q)]$, so $\mathbb{E}_\star[\ell(q)] = 0$.[21] Substituting the definition of J establishes (A.58).

For $\widehat{\pi}(q)$, the drift in (A.12) is bounded, as is the loading term $\beta\zeta q^\zeta \sigma^s(q)$. Consequently, the stochastic integral in the integrated equation for $\widehat{\pi}(q_t)$ is a square-integrable martingale on every finite horizon. Since $\widehat{\pi}$ itself is bounded, taking expectations of the integrated equation under invariant initialization gives

$$0 = T\mathbb{E}_\star\left[\rho\widehat{\pi}(q) - \kappa(q^{1+1/\eta} - 1) - \frac{\varepsilon - 1}{2}\widehat{\pi}(q)^2\right],$$

which establishes (A.59).

Equations (A.60), (A.61) and (A.62) are rearrangements of these identities. Since $J(q) = \phi_y(q-1) + (\phi_\pi - 1)\widehat{\pi}(q)$ and $r(q) = r^n + J(q)$,

$$\mathbb{E}_\star[J(q)] = \phi_y\mathbb{E}_\star[q-1] + (\phi_\pi - 1)\mathbb{E}_\star[\widehat{\pi}].$$

Equation (A.58) is the condition $\mathbb{E}_\star[\ell(q)] = 0$, written as

$$\mathbb{E}_\star[J(q)] + \sigma\mathbb{E}_\star[\sigma^s] + \frac{1}{2}\mathbb{E}_\star[(\sigma^s)^2] = 0.$$

When $\phi_y \neq 0$, solving for the mean output gap yields (A.62). The same condition implies $\mathbb{E}_\star[r^n - r(q)] = -\mathbb{E}_\star[J(q)]$, leading to (A.61). With $\sigma^s > 0$ in the interior and the invariant density strictly positive there, both $\mathbb{E}_\star[\sigma^s]$ and $\mathbb{E}_\star[(\sigma^s)^2]$ are strictly positive.

Equation (A.59) is linear in $\mathbb{E}_\star[\widehat{\pi}]$. Dividing by $\rho > 0$ and isolating this expectation yields (A.60). Finally, because $0 < \zeta < 1$, the function $q \mapsto q^\zeta$ is strictly concave. Since the invariant law is nondegenerate, strict Jensen's inequality gives $M_\zeta < M_1^\zeta$, establishing (A.63). □

A.7.3 Long-run distributions and sample paths

Proposition A.4 (Convergence without stabilization)**.** *From an arbitrary initial point $q_0 \in (0, \bar{q})$, expectations of bounded functions of the state converge to their invariant expectations. In particular,*

$$\lim_{t\to\infty} \mathbb{E}_{q_0}[q_t - 1] = M_1 - 1, \qquad \lim_{t\to\infty} \mathbb{E}_{q_0}[\widehat{\pi}(q_t)] = \underline{\widehat{\pi}} + \beta M_\zeta. \tag{A.65}$$

[21] This is an invariant-average restriction on the drift ℓ, not a pointwise condition requiring $\ell(q) = 0$ at every state.

For each bounded measurable state function f, its time average satisfies

$$\frac{1}{T}\int_0^T f(q_t)\,dt \longrightarrow \mathbb{E}_\star[f(q)] \quad \textit{almost surely.} \tag{A.66}$$

Nevertheless,

$$\liminf_{t\to\infty} q_t = 0, \qquad \limsup_{t\to\infty} q_t = \bar{q} \quad \textit{almost surely.} \tag{A.67}$$

Consequently, neither the output gap, centered inflation nor excess volatility has a pathwise limit.

Proof. Total-variation convergence in Proposition A.1 implies convergence of bounded expectations, which gives (A.65).

For (A.66), take the sampling interval $\Delta = 1$ in (A.44). The skeleton chain $(q_n)_{n\geq 0}$ then has the unique invariant probability $\varphi_\star(q)dq$, which is therefore ergodic (Hairer, 2018, Corollary 5.12). Let $g(q) = \int_0^1 \mathbb{E}_q[f(q_s)]ds$ and $Y_n = \int_n^{n+1} f(q_t)dt$. By invariance, $\mathbb{E}_\star[g(q)] = \mathbb{E}_\star[f(q)]$. By the Markov property, $\mathbb{E}[Y_n \mid \mathcal{F}_n] = g(q_n)$, so the differences $Y_n - g(q_n)$ form a martingale difference sequence bounded by $2\sup|f|$; applying the Azuma–Hoeffding inequality (Azuma, 1967) and the Borel–Cantelli lemma (Durrett, 2019), their averages converge to zero almost surely. Birkhoff's ergodic theorem (Hairer, 2018, Theorem 5.2) gives $N^{-1}\sum_{n<N} g(q_n) \to \mathbb{E}_\star[f(q)]$ almost surely, for $\varphi_\star$-almost every initial point. Because f is bounded, the fractional part of T contributes a term of order $1/T$, so (A.66) holds for $\varphi_\star$-almost every initial point, and in particular from some $q' \in (0, \bar{q})$. From an arbitrary q_0, the path reaches q' in finite time almost surely, as shown in the next paragraph, and by the strong Markov property the path after that time has the law of the process started at q'. The initial segment changes the time average by a term of order $1/T$, so (A.66) holds from every interior initial point.

For (A.67), fix $0 < a < c < \bar{q}$. For $q \in (a, \bar{q})$ and $c' \in (q, \bar{q})$, the exit time from (a, c') is finite almost surely by (A.23), and (A.80) gives $\mathbb{P}_q(\mathfrak{h}_a < \mathfrak{h}_{c'}) = (\mathfrak{s}(c') - \mathfrak{s}(q))/(\mathfrak{s}(c') - \mathfrak{s}(a))$. Letting $c' \uparrow \bar{q}$, where $\mathfrak{s}(c') \to +\infty$, gives $\mathbb{P}_q(\mathfrak{h}_a < \infty) = 1$. Symmetrically, letting the left barrier tend to zero, where $\mathfrak{s} \to -\infty$, gives $\mathbb{P}_q(\mathfrak{h}_c < \infty) = 1$ for every $q \in (0, c)$. By the strong Markov property, the path therefore visits a and c alternately infinitely often, almost surely, so $\liminf_t q_t \leq a$ and $\limsup_t q_t \geq c$. Applying this along sequences $a_n \downarrow 0$ and $c_n \uparrow \bar{q}$, and using $q_t \in (0, \bar{q})$ at all dates, proves (A.67). The gap and centered inflation are continuous, strictly increasing functions of q, so they oscillate between their bounds in (A.51). Excess volatility tends to $\bar{\sigma}^s > 0$ as $q \downarrow 0$ and to 0 as $q \uparrow \bar{q}$, so it also has no pathwise limit. □

A.8 An occasionally binding effective lower bound (ELB)

This section considers an alternative policy specification: the stochastic equilibrium of Lemma 2 survives a lower bound on the nominal policy rate. Throughout, let $\phi_y \geq 0$ and $\phi_\pi \geq 1$, not both

with equality. Let $i(q)$ denote the rate prescribed by the rule in (2.20) along the graph, defined in Section A.7, and let i^{elb} be a constant effective lower bound.[22] The effective policy rate is

$$i_t = \max\{i^{\text{elb}}, i(q_t)\}. \tag{A.68}$$

Along the graph, $i'(q) = \phi_y + \phi_\pi \widehat{\pi}'(q) > 0$. Hence any floor satisfying $i(0+) < i^{\text{elb}} < i(\bar{q})$ binds below a unique threshold $q_{\text{elb}} \in (0, \bar{q})$ and is slack above it.

The floor introduces a kink in the real-rate deviation $r_t - r^n$. Define

$$J_B(q) = i^{\text{elb}} - r^n - \bar{\pi} - \widehat{\pi}(q), \qquad J_{\text{elb}}(q) = \max\{J(q), J_B(q)\}, \tag{A.69}$$

with J in (A.3). The two functions J and J_B coincide at q_{elb}. Replacing J by J_{elb} in (A.4) gives

$$F_{\text{elb}}(q) = \frac{\omega}{\zeta} - \frac{\kappa}{\beta\zeta} q^{1+1/\eta-\zeta} - \frac{(\varepsilon-1)\beta}{2\zeta} q^{\zeta} - J_{\text{elb}}(q). \tag{A.70}$$

Equivalently, $F_{\text{elb}} = \min\{F, F_B\}$, where F_B is obtained by replacing J with J_B. We impose the additional restrictions

$$0 < \zeta < \frac{\varepsilon - 1}{2}, \qquad i(0+) < i^{\text{elb}} < i(\bar{q}). \tag{A.71}$$

Proposition A.5 (Stochastic solutions under an effective lower bound)**.** *Let $\phi_y \geq 0$ and $\phi_\pi \geq 1$, not both with equality, and assume that (A.2) and (A.71) hold. Then F_{elb} in (A.70) is continuous and strictly decreasing on $(0, \bar{q}]$, positive on $(0, \bar{q})$, and $F_{\text{elb}}(\bar{q}) = 0$.*

(i) Existence and equilibrium: *Let σ^s_{elb} be the volatility root in (3.6) with F replaced by F_{elb}. Then, the process (A.9), with J and σ^s replaced by J_{elb} and σ^s_{elb}, has a unique strong solution from $\forall q_0 \in (0, \bar{q})$. With $\pi_t = \bar{\pi} + \widehat{\pi}(q_t)$, it solves the Phillips curve (3.3b) and the IS equation with the policy rate (A.68), that is, (3.3a) with $J_{\text{elb}}(q_t)$ in place of $\phi_y(q_t - 1) + (\phi_\pi - 1)\widehat{\pi}_t$. Under passive supply in (3.4), it is a limiting equilibrium under the policy rule (A.68).*

(ii) Volatility and stationarity: *Excess volatility σ^s_{elb} is strictly positive on $(0, \bar{q})$. The conclusions of Proposition A.1 hold for this censored state equation, so both policy regimes have positive stationary probability.*

The proposition follows by checking monotonicity on the two pieces. On $(q_{\text{elb}}, \bar{q})$, $F_{\text{elb}} = F$, which is strictly decreasing by Lemma A.1. On the binding interval $(0, q_{\text{elb}})$,

$$F'_{\text{elb}}(q) = F'_B(q) = -\frac{\kappa(1 + 1/\eta - \zeta)}{\beta\zeta} q^{1/\eta-\zeta} + \beta\left(\zeta - \frac{\varepsilon - 1}{2}\right) q^{\zeta-1}. \tag{A.72}$$

Both terms of (A.72) are negative under (A.71). The two functions F_B and F coincide at the

[22] The zero lower bound is one such floor when $i(0+) < 0 < i(\bar{q})$.

threshold q_{elb}. The boundary classification is unchanged because the lower bound is slack near $\bar{q}$, while near $q \simeq 0$ it raises the real-rate deviation $r_t - r^n$ relative to the unconstrained rule, strengthening the drift of $\log q$ away from the origin.

Since the producers' pricing conditions contain no nominal interest rate, Lemma A.3 continues to establish the nonlinear Phillips curve for the family constructed from F_{elb}. The IS equation holds with $r_t - r^n = J_{\text{elb}}(q_t)$. In sum, the effective lower bound changes the policy-rate schedule and introduces a kink, but it leaves the upper endpoint and the continuum of stationary solutions indexed by (b, ζ) intact.

Figure A.1 illustrates the censored family under a zero lower bound, i.e., $i^{\text{elb}} = 0$. The graph $\widehat{\pi}(q)$ and the end point $\bar{q}$ are those of Figure 3. The floor truncates the policy rate on $(0, q_{\text{elb}})$, and the rule operates only on the narrow upper interval $(q_{\text{elb}}, \bar{q})$. On the binding interval, the floor lowers the forcing function F_{elb} and therefore excess volatility. Above q_{elb}, excess volatility coincides with the unconstrained one.

In a rigid-price economy, our earlier work (Dordal i Carreras and Lee, 2026) finds alternative zero-lower-bound equilibria with excess volatility when the central bank does not commit to future stabilization. If stabilization at a future date is certain, current excess volatility disappears and a unique equilibrium is selected. In the stationary economy constructed here, every future date repeats the same policy: the nominal rate follows the rule or binds at the effective lower bound, according to the current state. Under that recurrence, the stochastic solutions persist, and excess volatility remains strictly positive.

Proof of Proposition A.5. Differentiating the rule along the graph on $(0, \bar{q})$ gives

$$i'(q) = \phi_y + \phi_\pi \beta \zeta q^{\zeta - 1} > 0.$$

Hence $i(0+) < i(\bar{q})$, and a floor strictly between them is attained at exactly one $q_{\text{elb}} \in (0, \bar{q})$. Since $J(q) - J_B(q) = i(q) - i^{\text{elb}}$, for $q < q_{\text{elb}}$ we have $J_B(q) > J(q)$ and the floor binds, while for $q > q_{\text{elb}}$ we have $J_B(q) < J(q)$. At q_{elb}, $i(q_{\text{elb}}) = i^{\text{elb}}$ is $J(q_{\text{elb}}) = J_B(q_{\text{elb}})$, so J_{elb} and F_{elb} are continuous.

On $[q_{\text{elb}}, \bar{q}]$, $F_{\text{elb}} = F$. Lemma A.1 gives that this function is strictly decreasing, positive on $[q_{\text{elb}}, \bar{q})$ and zero at $\bar{q}$. On $(0, q_{\text{elb}})$, $J_{\text{elb}} = J_B$ and $J_B'(q) = -\widehat{\pi}'(q) = -\beta\zeta q^{\zeta - 1}$. Differentiating (A.70) therefore produces (A.72). The first term is negative because $\kappa > 0$, $\beta > 0$ and $1 + 1/\eta > \zeta$. The second term is also negative because $0 < \zeta < (\varepsilon - 1)/2$. Therefore F_B is strictly decreasing on $(0, q_{\text{elb}})$. Since $J_B > J$ there, $F_B < F$ on $(0, q_{\text{elb}})$, and the two functions coincide at q_{elb}. A continuous function that decreases strictly on each of $(0, q_{\text{elb}}]$ and $[q_{\text{elb}}, \bar{q}]$ decreases strictly on $(0, \bar{q}]$. Its only zero is $\bar{q}$, so $F_{\text{elb}} > 0$ on $(0, \bar{q})$. The volatility root with $F_{\text{elb}} > 0$ in place of F is therefore positive on the interior.

The resulting drift and diffusion agree with the unconstrained coefficients on $(q_{\text{elb}}, \bar{q})$. At the

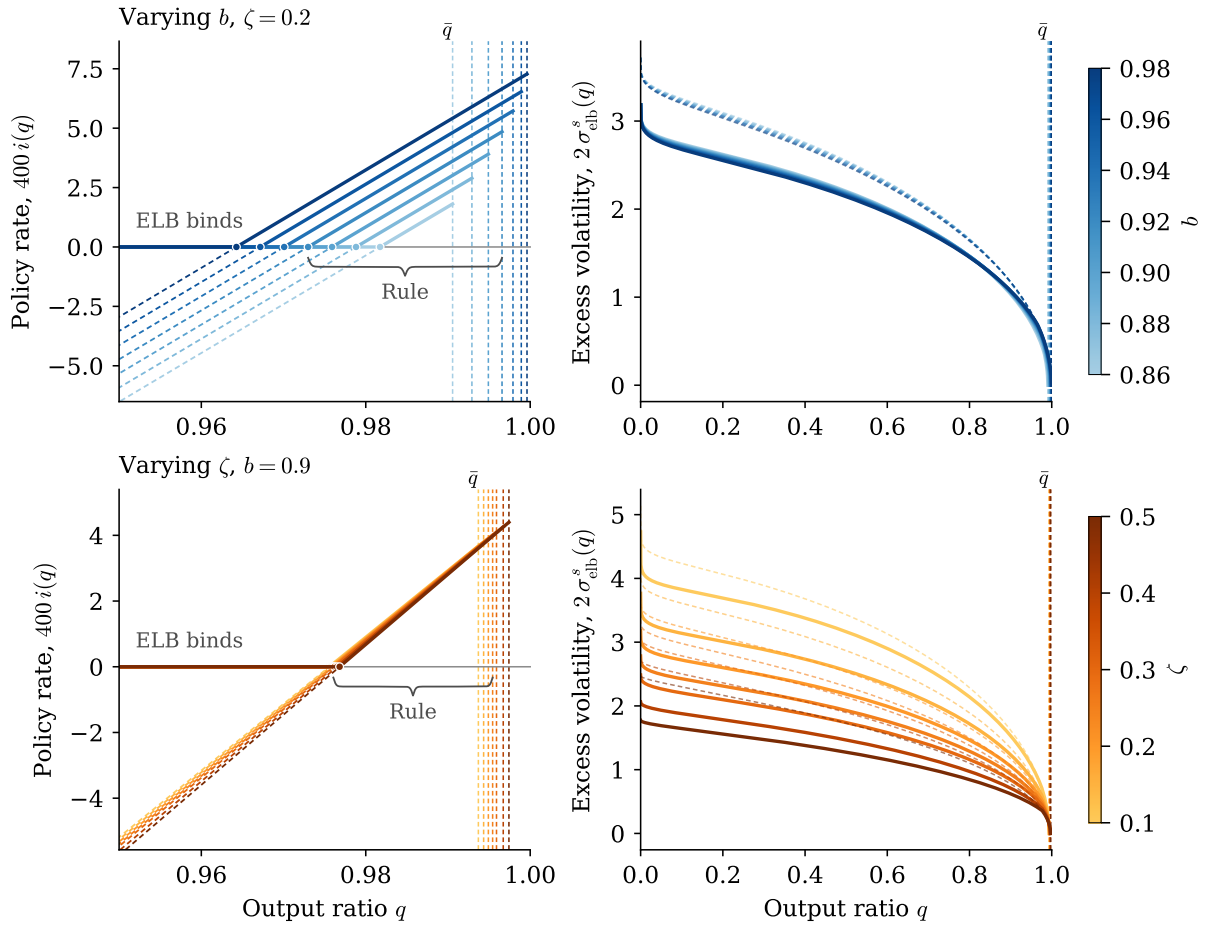


Figure A.1: Stochastic solutions under a zero lower bound. The left panels show the policy rate (A.68) for $q \in [0.95, 1]$: solid lines give the effective rate, dashed lines the unconstrained rule below the floor and dots the threshold q_{elb}. The braces mark $(q_{\text{elb}}, \bar{q})$ for the middle curve of each row. The right panels show excess volatility $\sigma^s_{\text{elb}}(q)$ (solid) and the unconstrained $\sigma^s(q)$ of (3.6) (dashed). Dashed vertical lines mark $\bar{q}$. The top row varies b with $\zeta = 0.2$; the bottom row varies ζ with $b = 0.9$. The calibration is that of Figure 3, with $b \geq 0.86$ so that $i(\bar{q}) > 0$; rates are annualized in percentage points and volatility is per square-root year. Every pair satisfies (A.2), (A.6) and (A.71).

single interior kink both one-sided derivatives are finite, so the coefficients are locally Lipschitz on $(0, \bar{q})$, and Section A.10 applies on each side of the kink, so they extend Lipschitz-continuously to $[0, \bar{q}]$. Near zero the floor binds, and

$$J_{\text{elb}}(0+) = i^{\text{elb}} - r^n - \bar{\pi} - \underline{\widehat{\pi}} = -\Phi_{\text{elb}}, \qquad \Phi_{\text{elb}} = r^n + \bar{\pi} + \underline{\widehat{\pi}} - i^{\text{elb}} < \Phi,$$

because $i^{\text{elb}} > i(0+) = r^n + \bar{\pi} + \underline{\widehat{\pi}} - \Phi$. Effectively, the real rate is closer to the natural rate than without the lower bound. In (A.16) and (A.17), this replaces Φ by the smaller shortfall Φ_{elb}. Strict monotonicity gives $F_{\text{elb}}(0+) = \omega/\zeta + \Phi_{\text{elb}} > 0$, and $\zeta^2 \Phi_{\text{elb}} < \omega$ because either $\Phi_{\text{elb}} \leq 0$ or $0 < \Phi_{\text{elb}} < \Phi$. The limiting log drift therefore remains strictly positive and points away from the origin. Near zero, $J_B(q) + \Phi_{\text{elb}} = -\beta q^\zeta = O(q^\zeta)$, so the lower-tail estimates of Section A.5 hold with the same exponent ϑ. The upper end-point is untouched, since the floor is slack on $(q_{\text{elb}}, \bar{q})$. Drift matching holds as in Lemma A.3, with J and F replaced by J_{elb} and F_{elb}, because their sum is unchanged and the Phillips curve contains no nominal rate.

The proof of Proposition A.1 uses the coefficients through their continuity, their end-point behavior and the scale function, all of which carry over: the scale integrand is continuous, so the scale function remains C^2. The only step that uses more smoothness is the characterization of the Girsanov bound of Section A.11.

Suppose the kink lies in the interval I used there, and let $x_k = x(q_{\text{elb}})$. The Lamperti map x has the absolutely continuous derivative $1/\upsilon$, with a bounded second derivative off q_{elb}, so Itô's formula for $x(q_t)$ holds, the time the path spends at q_{elb} having zero Lebesgue measure (Karatzas and Shreve, 1991, Section 3.7). The Lamperti drift $\mathfrak{b} = \mu/\upsilon - \upsilon'/2$ is bounded and piecewise continuously differentiable, with a jump $J_k = \mathfrak{b}(x_k+) - \mathfrak{b}(x_k-)$ at x_k, because υ' jumps there. Uniqueness in law of the stopped Lamperti equation follows from pathwise uniqueness of the Lipschitz state equation, because x is a bijection on I. The antiderivative $\mathfrak{B}$ of $\mathfrak{b}$ is continuous and twice continuously differentiable off x_k, where its derivative jumps by J_k. It is therefore a difference of convex functions, and the generalized Itô rule for Brownian motion (Karatzas and Shreve, 1991, Section 3.6) adds a local-time term to (A.90): on surviving paths,

$$\mathcal{E}_\Delta = \exp\left(\mathfrak{B}(x_\Delta) - \mathfrak{B}(x_0) - \frac{1}{2}\int_0^\Delta [\mathfrak{b}'(x_s) + \mathfrak{b}(x_s)^2] ds - \frac{1}{2} J_k L_\Delta^{x_k} \right),$$

where $\mathfrak{b}'(x_s)$ is the derivative off x_k and $L_\Delta^{x_k}$ is the local time of the reference Brownian motion at x_k, normalized by the occupation-density equation

$$\int_0^\Delta \psi(x_s) ds = \int_{\mathbb{R}} \psi(z) L_\Delta^z \, dz$$

for every bounded nonnegative measurable test function ψ. Hence, $\mathcal{E}_\Delta \geq \epsilon_{\mathcal{E}} \exp(-\frac{1}{2}|J_k| L_\Delta^{x_k})$ on surviving paths, with $\epsilon_{\mathcal{E}}$ as in (A.91) computed from the piecewise derivative. The local time is

not bounded pathwise, but its conditional mean is.

We make this bound explicit. Under the reference measure P, write the Brownian motion as $x_s = x + W_s$ and let

$$g_s(z) = \frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{z^2}{2s}\right)$$

denote its transition density over an interval of length s. Local time at x_k is the limit of time spent in a shrinking neighborhood of that point:

$$L_\Delta^{x_k} = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^\Delta 1_{\{|x_s - x_k| < \epsilon\}} ds. \tag{A.73}$$

Before taking the limit, write the finite-width approximation

$$L_{\Delta,\epsilon}^{x_k} = \frac{1}{2\epsilon} \int_0^\Delta 1_{\{|x_s - x_k| < \epsilon\}} ds. \tag{A.74}$$

For $z \in \mathbb{R}$, let $x_u^z = z + W_u$ be a Brownian motion restarted from z at its own time zero, and let $\mathbb{E}_{\mathrm{P}}^z$ denote expectation for that process. For any bounded nonnegative function h, the four equalities below record the passage from the approximation to an integral against the density of x_s:

$$\begin{aligned}
\mathbb{E}_{\mathrm{P}}\left[L_{\Delta,\epsilon}^{x_k} h(x_\Delta)\right] &= \frac{1}{2\epsilon} \mathbb{E}_{\mathrm{P}}\left[\int_0^\Delta 1_{\{|x_s - x_k| < \epsilon\}} h(x_\Delta) ds\right] \\
&= \frac{1}{2\epsilon} \int_0^\Delta \mathbb{E}_{\mathrm{P}}\left[1_{\{|x_s - x_k| < \epsilon\}} h(x_\Delta)\right] ds \\
&= \frac{1}{2\epsilon} \int_0^\Delta \int_{x_k - \epsilon}^{x_k + \epsilon} \mathbb{E}_{\mathrm{P}}\left[h(x_\Delta) \mid x_s = z\right] g_s(z - x)\, dz\, ds \\
&= \int_0^\Delta \frac{1}{2\epsilon} \int_{x_k - \epsilon}^{x_k + \epsilon} g_s(z - x)\, \mathbb{E}_{\mathrm{P}}^z\left[h(x_{\Delta - s}^z)\right] dz\, ds.
\end{aligned} \tag{A.75}$$

Note that the third equality conditions on the position $x_s = z$. The law of x_s starting from x has density $g_s(z - x)$, and the indicator keeps only those z that lie in $(x_k - \epsilon, x_k + \epsilon)$. The fourth is the Markov property. Conditional on $x_s = z$, the remaining path over an interval of length $\Delta - s$ has the same law as the restarted process x^z, so

$$\mathbb{E}_{\mathrm{P}}\left[h(x_\Delta) \mid x_s = z\right] = \mathbb{E}_{\mathrm{P}}^z\left[h(x_{\Delta - s}^z)\right].$$

The local-time approximation (A.74) converges to (A.73) in L^1, so $\mathbb{E}_{\mathrm{P}}\left[L_{\Delta,\epsilon}^{x_k} h(x_\Delta)\right]$ converges to $\mathbb{E}_{\mathrm{P}}[L_\Delta^{x_k} h(x_\Delta)]$. For $0 < s < \Delta$, the inner integral on the right-hand side of (A.75), divided by 2ϵ, is the average over $(x_k - \epsilon, x_k + \epsilon)$ of the continuous function

$$z \longmapsto g_s(z - x) \mathbb{E}_{\mathrm{P}}^z\left[h(x_{\Delta - s}^z)\right].$$

As $\epsilon \downarrow 0$, that average converges to the value at x_k. Consequently,

$$\begin{aligned}\mathbb{E}_{\mathrm{P}}\left[L_{\Delta}^{x_k} h(x_\Delta)\right] &= \int_0^{\Delta} g_s(x_k - x)\, \mathbb{E}_{\mathrm{P}}^{x_k}\left[h(x_{\Delta-s}^{x_k})\right] ds \\ &= \int_{\mathbb{R}} h(y) \left[\int_0^{\Delta} g_s(x_k - x) g_{\Delta-s}(y - x_k)\, ds\right] dy. \end{aligned} \tag{A.76}$$

Conditional on being at x_k, the density of ending at y after $\Delta - s$ is $g_{\Delta-s}(y - x_k)$. Integrating over all possible dates s at which local time is accumulated gives the second line. Since (A.76) holds for every such h, the joint measure of local time and the terminal state has density

$$\mathbb{E}_{\mathrm{P}}\left[L_{\Delta}^{x_k} 1_{\{x_\Delta \in dy\}}\right] = \left[\int_0^{\Delta} g_s(x_k - x) g_{\Delta-s}(y - x_k)\, ds\right] dy. \tag{A.77}$$

Substituting the Gaussian densities yields

$$\begin{aligned}&\int_0^{\Delta} g_s(x_k - x) g_{\Delta-s}(y - x_k)\, ds \\ &= \frac{1}{2\pi} \int_0^{\Delta} \frac{\exp\left[-\frac{(x_k - x)^2}{2s} - \frac{(y - x_k)^2}{2(\Delta - s)}\right]}{\sqrt{s(\Delta - s)}}\, ds \\ &\leq \frac{1}{2\pi} \int_0^{\Delta} \frac{ds}{\sqrt{s(\Delta - s)}} = \frac{1}{2\pi} \int_0^{1} \frac{du}{\sqrt{u(1 - u)}} = \frac{1}{2}. \end{aligned} \tag{A.78}$$

Thus the above bound is uniform in the starting point x, the kink x_k, the terminal point y and the horizon Δ.

It remains to condition on both the endpoint and survival in I. For $x, y \in x(C)$, the joint density of ending at y without exiting is

$$g_\Delta(y - x)\mathbb{P}_{\mathrm{P}}\big(\text{bridge from } x \text{ to } y \text{ stays in } I\big) \geq \epsilon_{\mathrm{G}} \epsilon_{\mathrm{br}}.$$

The corresponding local-time numerator is no larger than its unrestricted counterpart in (A.77), which is bounded by $\frac{1}{2}$ in (A.78). Taking the ratio therefore gives

$$\mathbb{E}_{\mathrm{P}}\left[L_{\Delta}^{x_k} \mid x_\Delta = y, \text{ no exit}\right] \leq \frac{1}{2\epsilon_{\mathrm{G}} \epsilon_{\mathrm{br}}} \qquad \text{for } x, y \in x(C).$$

Finally, the map $z \mapsto \exp(-|J_k|z/2)$ is convex. Jensen's inequality therefore gives

$$\mathbb{E}_{\mathrm{P}}\left[\mathcal{E}_\Delta \mid x_\Delta = y, \text{ no exit}\right] \geq \epsilon_{\mathcal{E}} \exp\left(-\frac{|J_k|}{4\epsilon_{\mathrm{G}} \epsilon_{\mathrm{br}}}\right) > 0.$$

This bound replaces $\epsilon_{\mathcal{E}}$ in (A.48), and the minorization (A.50) follows with a smaller positive constant. The remaining steps of the proof of Proposition A.1 apply unchanged to the censored state equation, and the strictly positive invariant density assigns positive probability to both

$(0, q_{\text{elb}})$ and $(q_{\text{elb}}, \bar{q})$. Under passive supply, money clearing restricts no allocation, so the argument of Section C.1 makes each such solution a limiting equilibrium under the policy rule (A.68). □

A.9 Scale, speed and boundary tests

Proposition A.1 states that the output ratio never leaves $(0, \bar{q})$, has the invariant density (A.14), and converges to the stationary distribution from every interior point. Section A.5 proves the statement, but three blocks of the proof rely on diffusion calculations that are not special to our model.

This section independently derives the scale function, the speed density, and the Kolmogorov forward equation. The proof evaluates the resulting integrals near $q = 0$ and $q = \bar{q}$, and uses the range of the scale function to pin down the invariant density. Section A.10 shows that the drift and diffusion of (A.9) extend Lipschitz-continuously to the closed interval, the input to the strong-existence part of the proof. Section A.11 constructs the unit-diffusion coordinate and the Girsanov weight used to minorize transition probabilities on a compact interior set. Each subsection explains the step of the proof that uses it.

The stationary density (A.14) is the normalized solution of the forward equation for (A.9). The boundary tests in Section A.5 are integrals of the scale density and the speed density of the same diffusion. Here we derive both densities, and the Kolmogorov forward equation, from basic Itô's formulas.[23]

The state follows (A.9) up to exit from $(0, \bar{q})$. For a twice continuously differentiable test function f, Itô's formula and the generator give

$$df(q_t) = \mathcal{L}f(q_t)dt + \upsilon(q_t)f'(q_t)dZ_t. \tag{A.79}$$

The generator $\mathcal{L}f = \mu f' + \upsilon^2 f''/2$ is the drift of $f(q_t)$. A scale function $\mathfrak{s}$ is a strictly increasing C^2 function that satisfies $\mathcal{L}\mathfrak{s} = 0$. Strict monotonicity makes $\mathfrak{s}$ a possible change of coordinate on the state interval: distinct output ratios receive distinct scale values, so the q-axis is merely relabeled.[24]

Substituting $f = \mathfrak{s}$ into (A.79) cancels the drift,

$$d\mathfrak{s}(q_t) = \upsilon(q_t)\mathfrak{s}'(q_t)dZ_t,$$

so $\mathfrak{s}(q_t)$ is a local martingale. Under the original coordinate the state has drift μ; in the scale coordinate the same path is driftless. This property yields the hitting probabilities below, which are ratios of scale distances. Section A.5 specializes these ratios to the endpoints of $(0, \bar{q})$.

Fix barriers $0 < a < q < c < \bar{q}$. Write $\mathfrak{h}_a$ for the first time the state equals a, starting

[23] The values of the endpoint integrals are computed in Section A.5, not here.
[24] The name scale refers to this relabeling.

from q, and $\mathfrak{h}_{a,c} = \mathfrak{h}_a \wedge \mathfrak{h}_c$. On the compact interval $[a, c]$ the processes $\mathfrak{s}(q_t)$ and $\upsilon(q_t)\mathfrak{s}'(q_t)$ are bounded up to $\mathfrak{h}_{a,c}$, so the stopped scale process is a bounded martingale. The diffusion is uniformly nondegenerate on $[a, c]$, so $\mathfrak{h}_{a,c}$ is finite almost surely; Section A.5 shows that it has finite expectation. Optional sampling at $\mathfrak{h}_{a,c}$ yields

$$\mathfrak{s}(q) = \mathfrak{s}(a)\mathbb{P}_q(\mathfrak{h}_a < \mathfrak{h}_c) + \mathfrak{s}(c)\mathbb{P}_q(\mathfrak{h}_c < \mathfrak{h}_a).$$

The two probabilities sum to one, so the hitting probability is a ratio of scale distances:

$$\mathbb{P}_q(\mathfrak{h}_a < \mathfrak{h}_c) = \frac{\mathfrak{s}(c) - \mathfrak{s}(q)}{\mathfrak{s}(c) - \mathfrak{s}(a)}. \tag{A.80}$$

If $\mathfrak{s}(a) \to -\infty$ as $a \downarrow 0$, then for each fixed interior c the probability of hitting a before c vanishes. If $\mathfrak{s}(c) \to +\infty$ as $c \uparrow \bar{q}$, the probability of hitting c before a fixed interior a likewise vanishes. Thus infinite scale distance from an interior point to an endpoint is sufficient for that endpoint to be unreachable in finite time. Section A.5 also proves non-exit independently, with a Lyapunov function.

The ratios (A.80) are evaluated once an explicit scale function is available. The equation $\mathcal{L}\mathfrak{s} = 0$ is the first-order differential equation $\mu\mathfrak{s}' + (\upsilon^2/2)\mathfrak{s}'' = 0$ for $\mathfrak{s}'$. Wherever $\mathfrak{s}' \neq 0$, it rearranges as

$$\frac{\mathfrak{s}''(q)}{\mathfrak{s}'(q)} = -\frac{2\mu(q)}{\upsilon(q)^2}.$$

The left side is $\partial_q \log \mathfrak{s}'$. Integrating from the interior reference $q_\circ$ in (A.13) and imposing $\mathfrak{s}'(q_\circ) = 1$ gives the unique positive scale density $\mathfrak{s}' = e^{-\mathcal{R}}$. The associated scale function, normalized by $\mathfrak{s}(q_\circ) = 0$, is

$$\mathfrak{s}'(q) = \exp\left[-\int_{q_\circ}^{q} \frac{2\mu(\tilde{q})}{\upsilon(\tilde{q})^2} d\tilde{q}\right], \qquad \mathfrak{s}(q) = \int_{q_\circ}^{q} \mathfrak{s}'(\tilde{q})\, d\tilde{q}. \tag{A.81}$$

Any other strictly increasing solution differs by a positive affine transformation, which cancels in (A.80). With this normalization, $\mathfrak{s}(a) \to -\infty$ is exactly $\int_0^{q_\circ} \mathfrak{s}' = \infty$, and $\mathfrak{s}(c) \to +\infty$ is $\int_{q_\circ}^{\bar{q}} \mathfrak{s}' = \infty$. Section A.5 verifies both.

The scale function does not record how long the path spends near each point: hitting concerns only the order of visits. An invariant probability, by contrast, depends on occupation: even if an endpoint is never reached, the process may linger nearby and pile up mass, or pass quickly and leave negligible mass.[25]

That weight is the speed measure, with Lebesgue density $\mathfrak{m}$. For a Borel set $A \subset (0, \bar{q})$, the integral $\int_A \mathfrak{m}(q)\, dq$ is the speed mass of A, and it is proportional to the expected calendar time

[25] Section A.5 uses this occupation weight to decide whether a path that starts arbitrarily close to an endpoint reaches an interior point in finite expected time.

the path spends in A. To obtain $\mathfrak{m}$, start from $\mathfrak{s}''/\mathfrak{s}' = -2\mu/\upsilon^2$. Then

$$\partial_q\left(\frac{1}{\mathfrak{s}'}\right) = \frac{2\mu}{\upsilon^2\mathfrak{s}'}.$$

For any C^2 test function f,

$$\partial_q\left(\frac{f'}{\mathfrak{s}'}\right) = \frac{1}{\mathfrak{s}'}\left(f'' + \frac{2\mu}{\upsilon^2}f'\right).$$

Multiply by $\upsilon^2\mathfrak{s}'/2$:

$$\frac{\upsilon^2\mathfrak{s}'}{2}\partial_q\left(\frac{f'}{\mathfrak{s}'}\right) = \mu f' + \frac{1}{2}\upsilon^2 f'' = \mathcal{L}f.$$

Define speed as

$$\mathfrak{m}(q) = \frac{2}{\upsilon(q)^2\mathfrak{s}'(q)}, \tag{A.82}$$

which gives

$$\mathcal{L}f = \frac{1}{\mathfrak{m}}\partial_q\left(\frac{f'}{\mathfrak{s}'}\right). \tag{A.83}$$

This is the scale-speed form of the generator. A similar identity holds in the coordinate $y = \mathfrak{s}(q)$. Set $f(q) = \hat{f}(y)$ and $S_t = \mathfrak{s}(q_t)$. Then $f' = \hat{f}'\mathfrak{s}'$, so $f'/\mathfrak{s}' = \hat{f}'$ and $\partial_q(f'/\mathfrak{s}') = \mathfrak{s}'\hat{f}''$. Therefore

$$\mathcal{L}f = \frac{\mathfrak{s}'}{\mathfrak{m}}\hat{f}''.$$

The coefficient equals the Itô generator of S. Indeed, $dS_t = \upsilon(q_t)\mathfrak{s}'(q_t)dZ_t$ and (A.82) give $\mathfrak{s}'/\mathfrak{m} = \frac{1}{2}(\upsilon\mathfrak{s}')^2$, so

$$\mathcal{L}^S\hat{f}(y) = \frac{1}{2}\left(\upsilon(q)\mathfrak{s}'(q)\right)^2\hat{f}''(y) = \frac{\mathfrak{s}'(q)}{\mathfrak{m}(q)}\hat{f}''(y).$$

Define $\mathfrak{m}^S$ by $\mathcal{L}^S\hat{f} = \hat{f}''/\mathfrak{m}^S$:

$$\frac{1}{\mathfrak{m}^S(y)} = \frac{\mathfrak{s}'(q)}{\mathfrak{m}(q)} = \frac{1}{2}\left(\upsilon(q)\mathfrak{s}'(q)\right)^2, \qquad \mathfrak{m}^S(y) = \frac{\mathfrak{m}(q)}{\mathfrak{s}'(q)} = \frac{2}{\left(\upsilon(q)\mathfrak{s}'(q)\right)^2}.$$

Hence, if a positive measure has Lebesgue density $\mathfrak{m}$ on the q-axis and $y = \mathfrak{s}(q)$, then $dy = \mathfrak{s}'(q)\,dq$ and

$$\int_A \mathfrak{m}(q)\,dq = \int_{\mathfrak{s}(A)} \mathfrak{m}^S(y)\,dy \tag{A.84}$$

for $A \subset (0, \bar{q})$. Both sides of (A.84) are speed mass, written in different coordinates.[26]

Scale removes drift. Speed restores the clock that this relabeling distorts: where $\upsilon\mathfrak{s}'$ is small,

[26] Neither $\mathfrak{m}$ nor $\mathfrak{m}^S$ integrates to one yet.

one unit of Brownian time in the y-coordinate consumes a large amount of calendar time, so $\mathfrak{m}$ is large.[27] To see this, note that by (A.79), $\mathcal{L}f(q)$ is the instantaneous mean change of $f(q_t)$ when the state equals q. If q_t has a Lebesgue density φ, then

$$\frac{d}{dt}\mathbb{E}[f(q_t)] = \int_0^{\bar{q}} (\mathcal{L}f)\,\varphi\,dq.$$

A stationary density makes the left side zero for every compactly supported f. By the scale-speed form (A.83), $(\mathcal{L}f)\,\mathfrak{m}$ is an exact q-derivative, so

$$\int_0^{\bar{q}} (\mathcal{L}f)\,\mathfrak{m}\,dq = \int_0^{\bar{q}} \partial_q\left(\frac{f'}{\mathfrak{s}'}\right) dq = 0$$

whenever f vanishes near the endpoints. Every multiple of $\mathfrak{m}$ therefore keeps these expectations constant. When $\int_0^{\bar{q}} \mathfrak{m}$ is finite, dividing by the total mass produces a probability density, and the probability of a Borel set A is the long-run fraction of calendar time the path spends in A.

Substituting the scale density (A.81) into (A.82) gives

$$\mathfrak{m}(q) = \frac{2}{\upsilon(q)^2 e^{-\mathcal{R}(q)}} = \frac{2e^{\mathcal{R}(q)}}{\upsilon(q)^2}.$$

The integrand of $\mathcal{N}$ in (A.13) is thus $\mathfrak{m}/2$. Finite total speed mass is the same statement as $\mathcal{N} < \infty$, and the factor two cancels when $\mathfrak{m}$ is normalized to a probability.[28]

This pairing shows that every multiple of $\mathfrak{m}$ is a stationary candidate. The forward equation then lists every feasible stationary density. It starts from the evolution of the law of q_t. Write $\varphi(t,q)$ for a smooth density of the current state on $(0,\bar{q})$. Then $\mathbb{E}[\mathcal{L}f(q_t)] = \int(\mathcal{L}f)\,\varphi\,dq$. Differentiating $\mathbb{E}[f(q_t)]$ in calendar time also gives $\int f\,\partial_t\varphi\,dq$. Integration by parts against compactly supported test functions converts the identity into the Kolmogorov forward, or Fokker–Planck, continuity equation

$$\partial_t\varphi(t,q) = -\partial_q \mathfrak{j}[\varphi](t,q), \qquad \mathfrak{j}[\varphi] = \mu\varphi - \frac{1}{2}\partial_q(\upsilon^2\varphi). \tag{A.85}$$

The current $\mathfrak{j}[\varphi]$ is the net probability flux through q. On the open interval, the identity describes the evolution of the law up to exit.

A stationary density is time-independent, so $\partial_q\mathfrak{j} = 0$ and the current equals a constant $\mathfrak{j}$. The stationary forward equation is therefore the first-order identity

$$\mu(q)\varphi(q) - \frac{1}{2}\partial_q(\upsilon(q)^2\varphi(q)) = \mathfrak{j}.$$

[27] For compactly supported C^2 test functions f and $\tilde{f}$ vanishing near the endpoints, $\int_0^{\bar{q}}(\mathcal{L}f)\,\tilde{f}\,\mathfrak{m}\,dq = \int_0^{\bar{q}} f\,(\mathcal{L}\tilde{f})\,\mathfrak{m}\,dq$.

[28] Infinite scale distance is sufficient for inaccessibility, and the entrance test is the nested integral of (A.81) and (A.82). Both belong to the Feller classification in Helland (1996, Sections 2 and 5, Theorem 8).

If the current vanishes, $\mathrm{j} = 0$, it rearranges as $\partial_q(\upsilon^2\varphi) = 2\mu\varphi$, or

$$\partial_q \log(\upsilon(q)^2\varphi(q)) = \frac{2\mu(q)}{\upsilon(q)^2}.$$

Integrating from $q_\circ$ gives $\upsilon^2\varphi \propto e^{\mathcal{R}}$. The zero-current member of the list is therefore a multiple of $e^{\mathcal{R}}/\upsilon^2$, hence a multiple of $\mathfrak{m}$.

For a nonzero constant current j, the same first-order equation is inhomogeneous. The zero-current solution $e^{\mathcal{R}}/\upsilon^2$ is already known, so write every candidate as that profile times an unknown function h of the state,

$$\varphi(q) = e^{\mathcal{R}(q)}\upsilon(q)^{-2}h(q).$$

Here h is the C^1 multiplier that remains after the homogeneous solution has been factored out. If h is constant, φ is a multiple of $\mathfrak{m}$. If h varies, it accounts for $\mathrm{j} \neq 0$. Differentiating the product $\upsilon^2\varphi = e^{\mathcal{R}}h$ and using $\partial_q\mathcal{R} = 2\mu/\upsilon^2$ gives

$$\partial_q(\upsilon^2\varphi) = e^{\mathcal{R}}\left(\frac{2\mu}{\upsilon^2}h + h'\right).$$

Substituting into $\mu\varphi - \partial_q(\upsilon^2\varphi)/2 = \mathrm{j}$, the terms that contain h cancel, and what remains is $-\frac{1}{2}e^{\mathcal{R}}h' = \mathrm{j}$. Hence $h' = -2\mathrm{j}\,e^{-\mathcal{R}}$. But $e^{-\mathcal{R}} = \mathfrak{s}'$, so $h' = -2\mathrm{j}\,\mathfrak{s}'$ and $h(q) = C_0 - 2\mathrm{j}\,\mathfrak{s}(q)$ for an integration constant C_0. Therefore $\varphi = (e^{\mathcal{R}}/\upsilon^2)(C_0 - 2\mathrm{j}\,\mathfrak{s})$. Since $\mathfrak{m} = 2e^{\mathcal{R}}/\upsilon^2$, this is

$$\varphi(q) = \mathfrak{m}(q)\big[C_{\mathrm{stat}} - \mathrm{j}\,\mathfrak{s}(q)\big], \tag{A.86}$$

with $C_{\mathrm{stat}} = C_0/2$. The speed density is positive on $(0, \bar{q})$, so the sign of φ is that of $C_{\mathrm{stat}} - \mathrm{j}\,\mathfrak{s}$. By construction $\mathfrak{s}(q_\circ) = 0$ and $\mathfrak{s}' > 0$, so $\mathfrak{s}$ is continuous and strictly increasing and takes every value between its endpoint limits. If those limits are $\mathfrak{s}(0+) = -\infty$ and $\mathfrak{s}(\bar{q}-) = +\infty$, a nonzero j makes $C_{\mathrm{stat}} - \mathrm{j}\,\mathfrak{s}$ change sign, and φ cannot stay nonnegative. Section A.5 verifies these limits, so only the zero-current solution, normalized speed, is viable.

These properties are the input to the boundary arguments of Section A.5. Infinite scale distance from an interior point makes the corresponding endpoint inaccessible. Finite speed mass near an endpoint makes its contribution to $\mathcal{N}$ finite, and therefore makes consumption utility integrable when log consumption varies with $\log q$. The nested integral of scale and speed is the entrance test: it diverges at $q = 0$ and converges at $q = \bar{q}$. After those integrals are in hand, the same proof returns to (A.86) and rules out a nonzero j, because the scale function tends to $-\infty$ at the origin and to $+\infty$ at $\bar{q}$.

A.10 Lipschitz extension of the state coefficients

The boundary classification in Section A.5 describes the behavior of the state process (A.9) near the endpoints 0 and $\bar{q}$. The recurrence argument there also applies Itô's formula to a Lyapunov function that diverges at both endpoints. Both steps use that the drift μ and the loading υ extend Lipschitz-continuously from $(0, \bar{q})$ to the compact interval $[0, \bar{q}]$. This subsection verifies that extension. The coefficients are C^∞ on the open interval, because F is smooth on $(0, 1)$ and the radicand $\sigma^2 + 2(1+\zeta)F$ exceeds σ^2 there.

The only obstruction is $q \downarrow 0$. Differentiating (A.4) gives (A.5):

$$F'(q) = -\frac{\kappa(1 + 1/\eta - \zeta)}{\beta\zeta} q^{1/\eta - \zeta} - \frac{(\varepsilon - 1)\beta}{2} q^{\zeta - 1} - \phi_y - (\phi_\pi - 1)\beta\zeta q^{\zeta - 1}.$$

The exponent $\zeta - 1$ is negative because $0 < \zeta < 1$. Thus $F'(q)$ may diverge as $q \downarrow 0$, and $(\sigma^s)'$ diverges with it. The state coefficients μ and υ, however, carry an extra factor q. Multiplying (A.5) by q gives

$$qF'(q) = -\frac{\kappa(1 + 1/\eta - \zeta)}{\beta\zeta} q^{1 + 1/\eta - \zeta} - \frac{(\varepsilon - 1)\beta}{2} q^{\zeta} - \phi_y q - (\phi_\pi - 1)\beta\zeta q^{\zeta}.$$

The shape conditions give $1 + 1/\eta - \zeta > 0$ and $\zeta > 0$, so every power on the right side tends to 0. Hence $qF'(q) \to 0$. The same holds for J: from (A.3), $J'(q) = \phi_y + (\phi_\pi - 1)\beta\zeta q^{\zeta - 1}$, and

$$qJ'(q) = \phi_y q + (\phi_\pi - 1)\beta\zeta q^{\zeta} \longrightarrow 0.$$

The volatility derivative follows from the chain rule through F. As in the lower-tail expansion of Section A.5, $d\sigma^s/dF = 1/(\sigma + (1+\zeta)\sigma^s)$, so

$$(\sigma^s)'(q) = \frac{F'(q)}{\sigma + (1+\zeta)\sigma^s(q)}, \qquad q(\sigma^s)'(q) = \frac{qF'(q)}{\sigma + (1+\zeta)\sigma^s(q)}.$$

The denominator is at least $\sigma > 0$, and the numerator tends to 0, so $q(\sigma^s)'(q) \to 0$. The function $(\sigma^s)'$ itself may be unbounded; only its product with q stays finite.

Now differentiate the state coefficients. Write $\mu(q) = q\big[J(q) + \sigma\sigma^s(q) + (\sigma^s(q))^2\big]$. The product rule gives

$$\mu'(q) = J(q) + \sigma\sigma^s(q) + (\sigma^s(q))^2 + qJ'(q) + \sigma\, q(\sigma^s)'(q) + 2\sigma^s(q)\, q(\sigma^s)'(q). \tag{A.87}$$

Send $q \downarrow 0$. The first three terms tend to $-\Phi + \sigma\bar{\sigma}^s + (\bar{\sigma}^s)^2$, which is finite. The remaining three each contain a factor qJ' or $q(\sigma^s)'$ and tend to 0, using $\sigma^s \to \bar{\sigma}^s$. Therefore $\mu'(0+)$ exists and equals $-\Phi + \sigma\bar{\sigma}^s + (\bar{\sigma}^s)^2$. For the loading $\upsilon(q) = q\sigma^s(q)$,

$$\upsilon'(q) = \sigma^s(q) + q(\sigma^s)'(q).$$

The first summand tends to $\bar{\sigma}^s$ and the second to 0, so $\upsilon'(0+) = \bar{\sigma}^s$. Both one-sided derivatives at 0 are finite.

At the upper endpoint no divergence appears. The root $\bar{q}$ lies in $(0,1)$, where F is C^∞, and $\sigma > 0$ keeps $\sqrt{\sigma^2 + 2(1+\zeta)F}$ smooth in a neighborhood of $\bar{q}$ inside $(0,1)$. Thus σ^s, μ and υ are C^1 up to $\bar{q}$. The derivatives μ' and υ' are continuous on $(0,\bar{q}]$ with finite limits at 0, hence bounded on $[0,\bar{q}]$, and the mean value theorem makes μ and υ Lipschitz there.

The proof of Proposition A.1 uses this extension twice: local strong existence and pathwise uniqueness run up to exit from $(0,\bar{q})$, and the Lyapunov function of the recurrence argument may be differentiated at states arbitrarily close to either endpoint. No boundary modification of the state equation is needed.

A.11 Lamperti coordinate and Girsanov comparison

Return to the recurrence argument of Section A.5, at the point where the Lyapunov sublevel C[29] has been enclosed in a strictly larger compact interval $I \subset (0,\bar{q})$. This subsection changes coordinates on I. In the new coordinate, Girsanov's theorem (Karatzas and Shreve, 1991) supplies the Radon–Nikodym derivative of the physical law with respect to Brownian motion, and that weight is bounded below on paths that remain in I.

On I, the loading υ is continuous and bounded away from zero, but it depends on q. The Lamperti coordinate (Iacus, 2008, Section 1.11.4) removes that state dependence. With the interior reference point $q_\circ$ of (A.13), set

$$x(q) = \int_{q_\circ}^{q} \upsilon(\tilde{q})^{-1}\, d\tilde{q},$$

which is finite for every $q \in (0,\bar{q})$ because υ is positive and continuous there. Then $x'(q) = 1/\upsilon(q)$ and $x''(q) = -\upsilon'(q)/\upsilon(q)^2$. On I, the map $q \mapsto x(q)$ is a C^2 diffeomorphism onto a compact interval I_x. Write $x(C) = \{x(q) : q \in C\}$ for the image of the sublevel. The map is continuous and strictly increasing, so $x(C)$ is a compact subinterval of the interior of I_x: the same set of states as C, read on the unit-diffusion axis rather than on the q-axis. Write $x_t = x(q_t)$. Itô's formula gives

$$dx_t = x'(q_t)\, dq_t + \frac{1}{2}x''(q_t)\upsilon(q_t)^2\, dt = \left(\frac{\mu(q_t)}{\upsilon(q_t)} - \frac{1}{2}\upsilon'(q_t)\right) dt + dZ_t.$$

The loading becomes identically one, and the remaining drift is a function of the current x-state. Define

$$\mathfrak{b}(x) = \frac{\mu(q)}{\upsilon(q)} - \frac{1}{2}\upsilon'(q),$$

[29] In Section A.5, $C = \{\mathcal{V} \le R_\mathcal{V}\}$ is a sublevel of the Lyapunov function (A.39). Properness of $\mathcal{V}$ makes it a compact subset of $(0,\bar{q})$.

evaluated at the unique $q \in I$ with $x(q) = x$. Smoothness of μ and υ on I makes $\mathfrak{b}$ continuously differentiable on I_x. Thus, while the path remains in I,

$$dx_t = \mathfrak{b}(x_t)\,dt + dZ_t, \tag{A.88}$$

Brownian motion plus a drift. With $\mathfrak{b}$ present, the transition density of (A.88) on I_x is not Gaussian, so its strict positivity is not yet known. With $\mathfrak{b} \equiv 0$, (A.88) is Brownian motion on I_x, whose transition density is Gaussian and strictly positive between interior points. Girsanov's theorem removes $\mathfrak{b}$ by reweighting the Brownian path measure; under the reweighted measure, paths follow (A.88).

Reweighting. Let W be a Wiener process on the x-axis. Define P as the law of $x_s = x_0 + W_s$ stopped at its exit time $\mathfrak{t}_I$ from I_x.[30] Define $\mathrm{Q}_{\mathrm{phys}}$ as the physical law of (A.88) stopped at the same exit. Girsanov's theorem reweights P so that the new paths solve (A.88), and uniqueness identifies the reweighted measure with $\mathrm{Q}_{\mathrm{phys}}$.

Under P, the process is $x_s = x_0 + W_s$ until exit.[31] On compact I_x, $\mathfrak{b}$ is bounded, so Novikov's condition (Novikov, 1972) holds under P, and by Karatzas and Shreve (1991, Corollary 3.5.13) the stopped exponential

$$\mathcal{E}_t = \exp\left(\int_0^{t\wedge\mathfrak{t}_I} \mathfrak{b}(x_s)\,dW_s - \frac{1}{2}\int_0^{t\wedge\mathfrak{t}_I} \mathfrak{b}(x_s)^2\,ds\right) \tag{A.89}$$

becomes a true martingale under P, with $\mathbb{E}_{\mathrm{P}}[\mathcal{E}_\Delta] = 1$. The quadratic correction in (A.89) is the Itô term that removes drift from $\mathcal{E}$ itself. Indeed, before exit, if $\mathcal{Y}_t = \int_0^t \mathfrak{b}\,dW - \frac{1}{2}\int_0^t \mathfrak{b}^2\,ds$, then $d\mathcal{Y} = \mathfrak{b}\,dW - \frac{1}{2}\mathfrak{b}^2\,dt$ and $(d\mathcal{Y})^2 = \mathfrak{b}^2\,dt$, so

$$d\mathcal{E}_t = e^{\mathcal{Y}_t}\,d\mathcal{Y}_t + \frac{1}{2}e^{\mathcal{Y}_t}(d\mathcal{Y}_t)^2 = \mathcal{E}_t\mathfrak{b}(x_t)\,dW_t.$$

Thus $\mathcal{E}$ is a local martingale under P.[32] Boundedness of $\mathfrak{b}$ upgrades it to a P-martingale. Define the measure Q by $d\mathrm{Q} = \mathcal{E}_\Delta\,d\mathrm{P}$; it is therefore a probability measure on path space.[33] The stopped density $\mathcal{E}_\Delta$ is defined on the full probability space; killing refers only to the paths retained when densities are computed.

The definition $d\mathrm{Q} = \mathcal{E}_\Delta\,d\mathrm{P}$ does not by itself declare any process to be a Brownian motion under Q; it only changes how paths are counted. Lévy's characterization (Karatzas and Shreve,

[30] This stopped path is the killed process of Section A.5: its position is no longer recorded after the first exit from I, or from its image I_x. The same exit stops $\mathrm{Q}_{\mathrm{phys}}$.

[31] The physical shock Z is not used on this space.

[32] It is not a local martingale under $\mathrm{Q}_{\mathrm{phys}}$.

[33] At this stage Q is only a constructed measure; it is identified with $\mathrm{Q}_{\mathrm{phys}}$ below.

1991, Theorem 3.3.16) is used to check a candidate. Write

$$\widehat{W}_t = W_t - \int_0^{t\wedge \mathfrak{t}_I} \mathfrak{b}(x_s)\, ds.$$

Quadratic variation is computed pathwise and does not depend on the measure. Subtracting a finite-variation integral does not change it, so $[\widehat{W},\widehat{W}]_t = [W,W]_t = t$ under both P and Q. Continuity is likewise pathwise. Therefore $\widehat{W}$ is a Q-Brownian motion as soon as it is a Q-local martingale.

On $[0,\Delta]$, a process N is a Q-local martingale if and only if $N\mathcal{E}$ is a P-local martingale. Here W is a P-Brownian motion and $d\mathcal{E}_t = \mathcal{E}_t \mathfrak{b}(x_t) 1_{\{t<\mathfrak{t}_I\}}\, dW_t$ under P. The product rule for $\widehat{W}\mathcal{E}$ gives

$$d(\widehat{W}_t\mathcal{E}_t) = \mathcal{E}_t(\widehat{W}_t \mathfrak{b}(x_t) 1_{\{t<\mathfrak{t}_I\}} + 1)\, dW_t,$$

so $\widehat{W}\mathcal{E}$ is a P-local martingale, and $\widehat{W}$ is a Q-local martingale. Combined with $[\widehat{W},\widehat{W}]_t = t$, Lévy's characterization shows that $\widehat{W}$ is a Q-Brownian motion.

Under P, $dx_t = dW_t$ until exit. Substituting $dW_t = d\widehat{W}_t + \mathfrak{b}(x_t)\, dt$ gives, under Q,

$$dx_t = \mathfrak{b}(x_t)\, dt + d\widehat{W}_t$$

until exit from I_x. Thus every Q-typical path solves the same stopped equation as the physical process (A.88). On the set I_x, the drift $\mathfrak{b}$ is C^1, hence Lipschitz. After extending $\mathfrak{b}$ to a bounded Lipschitz function on the real line, which does not affect paths before exit, (A.88) has pathwise uniqueness (Karatzas and Shreve, 1991, Theorem 5.2.5 and Remark 5.3.3) and hence uniqueness in law (Karatzas and Shreve, 1991, Proposition 5.3.20).[34] Thus $\mathrm{Q} = \mathrm{Q}_{\mathrm{phys}}$, so

$$\mathrm{Q}_{\mathrm{phys}}(A) = \mathbb{E}_{\mathrm{P}}\left[1_A \mathcal{E}_\Delta\right].$$

The Radon–Nikodym derivative $d\mathrm{Q}_{\mathrm{phys}}/d\mathrm{P}$ is the constructed factor $\mathcal{E}_\Delta$: the killed physical law is the killed Brownian law multiplied by $\mathcal{E}_\Delta$. The killed transition density of (A.88) then equals the killed Brownian (Gaussian) density times the conditional expectation of $\mathcal{E}_\Delta$ given the endpoints. A strictly positive lower bound on $\mathcal{E}_\Delta$ along surviving paths with endpoints in $x(C)$ therefore transfers a lower bound from the Brownian density to the drifted density.

Let $\mathfrak{B}$ be an antiderivative of $\mathfrak{b}$, so that $\mathfrak{B}' = \mathfrak{b}$ and $\mathfrak{B}'' = \mathfrak{b}'$. On a surviving reference path and under P, Itô's formula applied to $\mathfrak{B}(x_s)$, with $dx_s = dW_s$ and $(dx_s)^2 = ds$, gives

$$d\mathfrak{B}(x_s) = \mathfrak{B}'(x_s)\, dx_s + \frac{1}{2}\mathfrak{B}''(x_s)\, ds = \mathfrak{b}(x_s)\, dW_s + \frac{1}{2}\mathfrak{b}'(x_s)\, ds.$$

[34] Under uniqueness in law, any two measures on path space under which the process solves the same equation from the same initial value coincide.

Integrate from 0 to Δ and solve for the stochastic integral:

$$\int_0^\Delta \mathfrak{b}(x_s)\, dW_s = \mathfrak{B}(x_\Delta) - \mathfrak{B}(x_0) - \frac{1}{2}\int_0^\Delta \mathfrak{b}'(x_s)\, ds.$$

Substitute this identity into the exponent of $\mathcal{E}_\Delta$:

$$\int_0^\Delta \mathfrak{b}\, dW - \frac{1}{2}\int_0^\Delta \mathfrak{b}^2\, ds = \mathfrak{B}(x_\Delta) - \mathfrak{B}(x_0) - \frac{1}{2}\int_0^\Delta [\mathfrak{b}'(x_s) + \mathfrak{b}(x_s)^2] ds.$$

Therefore

$$\mathcal{E}_\Delta = \exp\left(\mathfrak{B}(x_\Delta) - \mathfrak{B}(x_0) - \frac{1}{2}\int_0^\Delta [\mathfrak{b}'(x_s) + \mathfrak{b}(x_s)^2] ds\right). \tag{A.90}$$

On paths that start and end in $x(C)$ and never leave I_x, the likelihood $\mathcal{E}_\Delta$ in (A.90) is bounded below by a positive constant $\epsilon_{\mathcal{E}}$. Indeed, $\mathfrak{B}$ is continuous on the compact I_x, so $\mathfrak{B}(x_\Delta) - \mathfrak{B}(x_0) \geq -2\sup_{I_x}|\mathfrak{B}|$. Also, $\mathfrak{b}' + \mathfrak{b}^2$ is continuous on I_x, hence bounded in absolute value by some finite $M_{\mathfrak{b}}$. The time integral over an interval of length Δ is therefore at most $M_{\mathfrak{b}}\Delta$, and[35]

$$\mathcal{E}_\Delta \geq \exp(-2\sup_{I_x}|\mathfrak{B}| - M_{\mathfrak{b}}\Delta/2) =: \epsilon_{\mathcal{E}} > 0. \tag{A.91}$$

The proof in Section A.5 uses this bound in (A.48): it multiplies $\epsilon_{\mathcal{E}}$ by the lower bound on the killed Brownian density between endpoints in $x(C)$, changes variables back to the output ratio, and obtains the minorization constant on C.

B Individual admissibility and verification

This section establishes global individual optimality in two steps. First, Sections B.1 and B.2 are general, defining the admissible plans of households and producers in the economy of Section 2 and stating sufficient conditions under which the first-order conditions of Appendices A and B are globally optimal among those plans. Section B.3 then verifies those conditions for the limiting equilibria of Definition 2: the stochastic solutions of Lemma 2 and the fully stabilized allocation.[36]

B.1 Household optimality

The first-order conditions of Appendix A are necessary conditions. Household plans are restricted to a domain of admissible plans defined by discounted gross-flow integrability (B.1a) and expected

[35] The bound depends on C and I only through the compact interval I_x.

[36] Note that the admissible plans restrict individual deviations as part of the household and producer problems; they are not inferred from stationarity or market clearing.

solvency (B.1b):[37]

$$\mathbb{E}_t \int_t^\infty \frac{\xi_s}{\xi_t}\left[C_s^{\text{dev}} + \frac{w_s}{p_s}L_s^{\text{dev}} + \delta_s m_s^{\text{dev}}\right] ds < \infty, \tag{B.1a}$$

$$\liminf_{T\to\infty} \mathbb{E}_t\left[\frac{\xi_T}{\xi_t} a_T^{\text{dev}}\right] \geq 0, \tag{B.1b}$$

where $\xi_t = e^{-\rho t}/Q_t$. Starting from date t, an admissible deviation obeys the cash constraint (2.4) and the budget constraint (A.1), which holds before money balances are set equal to X_t^h/v. It is also locally integrale. These restrictions apply conditionally from every decision date to the candidate and its admissible alternatives. Alternatives start from the same inherited wealth, face the same prices and lump-sum transfers, and preserve the information and funding restrictions of Section 2.2.

Consider a symmetric candidate satisfying the budget in (2.11), the expenditure-minimizing allocation of Appendix A.2, the intratemporal labor condition (2.12) and the Euler equation (A.3). Let $\mathcal{I}_s = D_s/p_s + \mathcal{T}_s$ denote its given nonlabor income and U_s its flow utility. Define the real money-market account and its normalized marginal-utility pricing process by

$$\mathcal{B}_s^r = \exp\left(\int_0^s r_{\tilde{s}} d\tilde{s}\right), \qquad \mathcal{D}_{t,s} = \frac{\xi_s \mathcal{B}_s^r}{\xi_t \mathcal{B}_t^r}, \qquad s \geq t. \tag{B.2}$$

Note in (B.2) that the account compounds at the real rate, not the nominal rate.

Proposition B.1 (Household optimality—sufficiency)**.** *For each decision date t, suppose that $\mathcal{D}_{t,s}$ is a martingale on every finite interval $[t,T]$, that the candidate satisfies (B.1a), and that*

$$\mathbb{E}_t \int_t^\infty \frac{\xi_s}{\xi_t}|\mathcal{I}_s| ds < \infty, \qquad \mathbb{E}_t \int_t^\infty e^{-\rho(s-t)}|U_s| ds < \infty, \tag{B.3}$$

$$\lim_{T\to\infty} \mathbb{E}_t\left[\frac{\xi_T}{\xi_t} a_T\right] = 0. \tag{B.4}$$

Then the candidate maximizes expected discounted utility over the admissible plans.

Proof. For an alternative, put $\mathcal{J}_s^{\text{dev}} = (w_s/p_s)L_s^{\text{dev}} + \mathcal{I}_s - C_s^{\text{dev}} - \delta_s m_s^{\text{dev}}$. The budget (A.1) gives

$$a_T^{\text{dev}} = \frac{\mathcal{B}_T^r}{\mathcal{B}_t^r} a_t + \int_t^T \frac{\mathcal{B}_T^r}{\mathcal{B}_s^r} \mathcal{J}_s^{\text{dev}} ds.$$

For $t \leq s \leq T$, the martingale condition gives $\mathbb{E}_s[\xi_T \mathcal{B}_T^r] = \xi_s \mathcal{B}_s^r$, leading to

$$\mathbb{E}_t\left[\frac{\xi_T}{\xi_t} a_T^{\text{dev}}\right] = a_t + \mathbb{E}_t \int_t^T \frac{\xi_s}{\xi_t} \mathcal{J}_s^{\text{dev}} ds. \tag{B.5}$$

[37] Condition (B.1a) requires the expected present value of the deviation's consumption, labor income and carrying cost of money, discounted with ξ, to be finite. (B.1b) requires the expected present value of the deviation's wealth to be asymptotically nonnegative, ruling out Ponzi schemes.

The same equality holds for the candidate. The deviation's solvency condition (B.1b) and the candidate equality (B.4) imply

$$\mathbb{E}_t \int_t^\infty \frac{\xi_s}{\xi_t} \left[C_s^{\text{dev}} + \delta_s m_s^{\text{dev}} - Q_s - \frac{w_s}{p_s}(L_s^{\text{dev}} - L_s) \right] ds \leq 0. \tag{B.6}$$

The expenditure bound in Appendix A.2, concavity of log basket utility and convexity of labor disutility give the pointwise inequality

$$\begin{aligned} U_s^{\text{dev}} - U_s &\leq \frac{C_s^{\text{dev}} + \delta_s m_s^{\text{dev}} - Q_s}{Q_s} - L_s^{1/\eta}(L_s^{\text{dev}} - L_s) \\ &= \frac{1}{Q_s} \left[C_s^{\text{dev}} + \delta_s m_s^{\text{dev}} - Q_s - \frac{w_s}{p_s}(L_s^{\text{dev}} - L_s) \right]. \end{aligned} \tag{B.7}$$

where we use the intratemporal condition (2.12). We them multiply (B.7) by $e^{-\rho(s-t)}$ and take the conditional expectation of the integral. The definition $\xi_s = e^{-\rho s}/Q_s$ gives $\xi_s/\xi_t = e^{-\rho(s-t)}(Q_t/Q_s)$, so

$$e^{-\rho(s-t)} \frac{1}{Q_s} = \frac{1}{Q_t} \frac{\xi_s}{\xi_t}. \tag{B.8}$$

The right-hand side of the discounted utility comparison is therefore $(1/Q_t)$ times the integrand of (B.6). Since $Q_t > 0$ is known at date t, (B.6) yields

$$\mathbb{E}_t \int_t^\infty e^{-\rho(s-t)} (U_s^{\text{dev}} - U_s) ds \leq 0. \tag{B.9}$$

The upper bound is integrable by the gross-flow restrictions (B.1a). Since candidate absolute utility is integrable, the alternative's utility is integrable as well. (B.9) establishes the sufficient condition. □

B.2 Producer optimality

Consider a symmetric candidate solution $\Omega_s = 1$, $u_s = \widehat{\pi}_s$ satisfying (B.3), with costate λ_s and local martingale N_s. Write $\bar{f}_s = f_s(1, \widehat{\pi}_s)$, where f_s is the normalized payoff in (B.2), and define

$$\mathfrak{k}_s = \left(\frac{1}{\alpha} - 1 \right) L_s^{1+1/\eta} \geq 0. \tag{B.10}$$

This is the intercept of the concave payoff's supporting plane at the symmetric candidate.

Proposition B.2 (Producer optimality)**.** *Suppose that, conditionally from each decision date t,*

the candidate satisfies

$$\lambda_s \geq 0 \quad (s \geq t), \qquad \mathbb{E}_t \int_t^\infty e^{-\rho(s-t)}(|\bar{f}_s| + \mathfrak{k}_s)ds < \infty, \tag{B.11}$$

$$\lim_{T\to\infty} \mathbb{E}_t[e^{-\rho(T-t)}\lambda_T] = 0, \tag{B.12}$$

and that $\int_t^s e^{-\rho(\tilde{s}-t)}dN_{\tilde{s}}$ *is a true martingale on every finite horizon. Then the candidate is optimal among all admissible pricing plans with the same inherited individual price, hence* $\Omega_t^j = 1$.

Proof. Recall the normalized payoff of Appendix B.1,

$$f_s(\Omega, u) = \frac{\Omega}{H_{\gamma,s}} - L_s^{1+1/\eta}\Omega^{1/\alpha} - \frac{\tau}{2H_{\gamma,s}}\frac{u^2}{\Omega}. \tag{B.13}$$

The power $\Omega^{1/\alpha}$ and the perspective u^2/Ω are convex on $\Omega > 0$. Their first-order supporting planes at the above symmetric point $(\Omega, u) = (1, \widehat{\pi}_s)$ are

$$\Omega^{1/\alpha} \geq 1 + \frac{\Omega - 1}{\alpha}, \tag{B.14}$$

$$\frac{u^2}{\Omega} \geq \widehat{\pi}_s^2 - \widehat{\pi}_s^2(\Omega - 1) + 2\widehat{\pi}_s(u - \widehat{\pi}_s), \tag{B.15}$$

and the exact gap in (B.15) equals $(u - \widehat{\pi}_s\Omega)^2/\Omega$. Thus, (B.13) yields the nonnegative remainder

$$\mathfrak{R}_s(\Omega, u) = L_s^{1+1/\eta}\left[\Omega^{1/\alpha} - 1 - \frac{\Omega - 1}{\alpha}\right] + \frac{\tau}{2H_{\gamma,s}\Omega}(u - \widehat{\pi}_s\Omega)^2 \geq 0. \tag{B.16}$$

Equivalently,

$$\begin{aligned} f_s(\Omega, u) + \mathfrak{R}_s(\Omega, u) = \frac{\Omega}{H_{\gamma,s}} &- L_s^{1+1/\eta}\left[1 + \frac{\Omega - 1}{\alpha}\right] \\ &- \frac{\tau}{2H_{\gamma,s}}[\widehat{\pi}_s^2 - \widehat{\pi}_s^2(\Omega - 1) + 2\widehat{\pi}_s(u - \widehat{\pi}_s)]. \end{aligned} \tag{B.17}$$

Collecting the linear terms in (Ω, u) and using

$$f_{s,\Omega}(1, \widehat{\pi}_s) = \frac{1}{H_{\gamma,s}} - \frac{1}{\alpha}L_s^{1+1/\eta} + \frac{\tau\widehat{\pi}_s^2}{2H_{\gamma,s}} \tag{B.18}$$

gives

$$f_s(\Omega, u) + \mathfrak{R}_s(\Omega, u) = f_{s,\Omega}(1, \widehat{\pi}_s)\,\Omega - \frac{\tau\widehat{\pi}_s}{H_{\gamma,s}}u + \mathfrak{k}_s. \tag{B.19}$$

At the symmetric equilibrium, the optimal control (B.3) is $\lambda_s = -\tau\widehat{\pi}_s/[(\varepsilon-1)H_{\gamma,s}]$, so $(\varepsilon-1)\lambda_s =$

$-\tau\widehat{\pi}_s/H_{\gamma,s}$. Substituting that identity into (B.19) yields

$$f_s(\Omega,u) = f_{s,\Omega}(1,\widehat{\pi}_s)\,\Omega + (\varepsilon-1)\lambda_s u + \mathfrak{k}_s - \mathfrak{R}_s(\Omega,u). \tag{B.20}$$

The state equation (B.1) and costate equation (B.3) then give, for any admissible alternative,

$$d\left(e^{-\rho(s-t)}\lambda_s\Omega_s^j\right) = e^{-\rho(s-t)}\left[\mathfrak{k}_s - f_s(\Omega_s^j,u_s^j) - \mathfrak{R}_s(\Omega_s^j,u_s^j)\right]ds + e^{-\rho(s-t)}\Omega_s^j\,dN_s. \tag{B.21}$$

Denote $f_s^+ = \max\{f_s(\Omega_s^j,u_s^j),0\}$ and $f_s^- = \max\{-f_s(\Omega_s^j,u_s^j),0\}$. Then for stopping times $\mathfrak{t}_n \uparrow \infty$ and every finite T, (B.21) leads to

$$\begin{aligned}\mathbb{E}_t[e^{-\rho(T-t)}\lambda_T\Omega_T^j] + \mathbb{E}_t\int_t^T e^{-\rho(s-t)}[f_s^+ + \underbrace{\mathfrak{R}_s(\Omega_s^j,u_s^j)}_{\geq 0}]ds \\ \leq \lambda_t + \mathbb{E}_t\int_t^T e^{-\rho(s-t)}[\mathfrak{k}_s + \underbrace{f_s^-}_{\geq 0}]ds,\end{aligned} \tag{B.22}$$

which with (B.11) yields

$$\mathbb{E}_t\int_t^\infty e^{-\rho(s-t)}f_s(\Omega_s^j,u_s^j)ds \leq \lambda_t + \mathbb{E}_t\int_t^\infty e^{-\rho(s-t)}\mathfrak{k}_s ds. \tag{B.23}$$

For the candidate, $\Omega_s = 1$ and the remainder vanishes. Finite-horizon martingality permits expectations in (B.21) without stopping. Candidate integrability and (B.12) then give

$$\mathbb{E}_t\int_t^\infty e^{-\rho(s-t)}\bar{f}_s ds = \lambda_t + \mathbb{E}_t\int_t^\infty e^{-\rho(s-t)}\mathfrak{k}_s ds.$$

The candidate attains the bound and is globally optimal on the stated domain. □

B.3 Verification for the limiting equilibria

Fix a shape pair satisfying (A.2) and (A.6) and the interior solution from Proposition A.1. We verify the hypotheses of Propositions B.1 and B.2 for the limiting allocation, using the household and producer deviation domains of Sections B.1 and B.2.

Allocation and financial positions. In the cashless limit $H_{\gamma,t} = 1$, $Q_t = C_t = Y_t = A_t L^n q_t$ and $L_t = L^n q_t$. The labor condition and the rebate of producer fees imply

$$\frac{w_t}{p_t} = C_t L_t^{1/\eta}, \qquad \frac{D_t}{p_t} = C_t - \frac{w_t}{p_t}L_t. \tag{B.24}$$

The issuer's balance sheet and market clearing (Sections 2.5 and 2.6) give $K_t = M_t^{\mathrm{s}}$ and household positions $B_t^h = -K_t$, $M_t^h = M_t^{\mathrm{s}}$. Thus candidate net wealth is $a_t^h = 0$ before taking the limit as

well as in it. Actual money, loans, carrying costs and issuer transfers vanish in the limit; balances measured per unit of cash share $1-\gamma$ need not. In particular, $\mathcal{I}_t = D_t/p_t + \mathcal{T}_t = C_t - (w_t/p_t)L_t$ in the limiting budget, so zero net wealth satisfies that budget. Purchases and balances have the feasible implementation described in Section 2.2.

Household integrability, pricing and transversality. The financial restrictions are (B.1a)–(B.1b); the candidate terminal equality is (B.4). Since $0 < L_t < L^n\bar{q}$,

$$\frac{C_s + (w_s/p_s)L_s}{Q_s} = 1 + L_s^{1+1/\eta}, \qquad \frac{|\mathcal{I}_s|}{Q_s} \leq 1 + L_s^{1+1/\eta}. \tag{B.25}$$

Consequently, from every decision date t,

$$\mathbb{E}_t \int_t^\infty \frac{\xi_s}{\xi_t}\left[C_s + \frac{w_s}{p_s}L_s + \delta_s m_s\right] ds \leq \frac{C_t}{\rho}\left[1 + (L^n\bar{q})^{1+1/\eta}\right] < \infty. \tag{B.26}$$

The same bound applies to the expected present value of $|\mathcal{I}_s|$. The absolute-utility estimate in Section A.6, applied conditionally from t, verifies the other part of (B.3). Candidate wealth is identically zero, so both solvency and (B.4) hold exactly. Solvency for alternative plans remains an admissibility restriction.

The IS equation gives expenditure loading $\sigma + \sigma^s(q_s)$ and the Euler drift for $\xi_s = e^{-\rho s}/C_s$. Thus the normalized process in (B.2) is

$$\mathcal{D}_{t,s} = \exp\left\{-\int_t^s [\sigma + \sigma^s(q_{\tilde{s}})]dZ_{\tilde{s}} - \frac{1}{2}\int_t^s [\sigma + \sigma^s(q_{\tilde{s}})]^2 d\tilde{s}\right\}. \tag{B.27}$$

The loading is bounded by $\sigma + \sup \sigma^s$, which is finite because σ^s extends continuously to $[0, \bar{q}]$ with the finite limit (A.16). Novikov's condition (Novikov, 1972) holds on every finite horizon, and from Karatzas and Shreve (1991), (B.27) is a true martingale. Proposition B.1 now applies: the household allocation is globally optimal among alternatives satisfying the original gross-flow and solvency restrictions, with the same inherited wealth and given prices and transfers.

Producer admissibility and transversality. The candidate's own inflation equals $\bar{\pi} + \widehat{\pi}(q_s)$, a bounded adapted process, so it is an admissible locally square-integrable control in Appendix B.1. At the symmetric price, the normalized profit, supporting-plane intercept and costate are

$$\bar{f}_s = 1 - L_s^{1+1/\eta} - \frac{\tau}{2}\widehat{\pi}(q_s)^2, \qquad \mathfrak{k}_s = \left(\frac{1}{\alpha} - 1\right) L_s^{1+1/\eta}, \tag{B.28}$$

$$\lambda_s = -\frac{\tau}{\varepsilon - 1}\widehat{\pi}(q_s), \qquad 0 < \lambda_s \leq -\frac{\tau}{\varepsilon - 1}\underline{\widehat{\pi}}. \tag{B.29}$$

Both flow terms are bounded. They therefore satisfy (B.11), including finite absolute candidate profit. The costate bound gives the terminal condition (B.12), since

$$0 \leq \mathbb{E}_t[e^{-\rho(T-t)}\lambda_T] \leq -\frac{\tau\widehat{\pi}}{\varepsilon-1}e^{-\rho(T-t)} \longrightarrow 0.$$

Lemma A.3 and the price-setting derivation in Appendix B.2 verify the costate equation. Its local-martingale part has bounded loading,

$$dN_s = -\frac{\tau\beta\zeta}{\varepsilon-1}q_s^{\zeta}\sigma^s(q_s)dZ_s. \tag{B.30}$$

The discounted integral of dN_s is therefore a true martingale on each finite horizon. All hypotheses of Proposition B.2 hold, so the symmetric price is globally optimal against its full signed-payoff deviation domain.

The same criterion verifies the fully stabilized limiting allocation: labor becomes the constant L^n, the expenditure loading is σ, and the producer costate and its martingale part are zero.

C Money supply and equilibrium selection

Let $\tilde{d}_t = 1/[v(1+\delta_t/v)^{\theta}] > 0$ be the limiting demand coefficient in (3.1), where $v > 0$, $\theta > 1$ and the nonnegative spread process δ_t are given. Recall that real money is measured per unit of cash share. Limiting money-market clearing is

$$\tilde{m}_t^{\mathrm{s}} = \tilde{d}_t Y_t^n q_t. \tag{C.1}$$

Together with the IS equation, the Phillips curve and the policy rule in (3.3a), (3.3b) and (2.20), and with the chosen money rule and its compatible initial stock, this is the limiting equilibrium system. Output, consumption and labor are $Y_t = C_t = Y_t^n q_t$ and $L_t = L^n q_t$, and the price level is

$$p_t = p_0 \exp\Big(\int_0^t [\bar{\pi} + \widehat{\pi}(q_s)]ds\Big)$$

along the stochastic construction.

C.1 Proof of Proposition 1 (passive supply)

Proof. Under passive supply, $\tilde{m}_t^{\mathrm{s}} = \tilde{d}_t Y_t$, money clearing is an identity, including at date zero. At the fully stabilized allocation $q_t = 1$, $\widehat{\pi}_t = 0$ and $\sigma_t^s = \sigma_t^{\pi} = 0$, both drifts in (3.3a) and (3.3b) vanish. The policy rule reduces to $i_t = r^n + \bar{\pi}$, and passive money supply guarantees the initial stock $\tilde{m}_0 = \tilde{d}_0 Y_0^n$.

Now fix any pair (b, ζ) satisfying (A.2) and (A.6) and any $q_0 \in (0, \bar{q})$. Proposition A.1 supplies

the unique strong solution of (A.9) for all dates, with interior paths. Set

$$\pi_t = \bar{\pi} + \widehat{\pi}(q_t), \qquad \sigma_t^s = \sigma^s(q_t), \qquad \sigma_t^\pi = \beta\zeta q_t^\zeta \sigma^s(q_t),$$

and define i_t by the policy rule (2.20). With these, Lemma A.3 verifies the IS equation (3.3a) and Phillips curve (3.3b). Passive money $\tilde{m}_t^{\mathrm{s}} = \tilde{d}_t Y_t^n q_t$ clears the market and supplies the compatible initial stock. The graph and policy functions are bounded on the state interval, so the price level is positive and finite on every finite horizon. All processes are adapted and have the stipulated continuity.

Thus, the same interest-rate rule and passive supply support the fully stabilized allocation and the stochastic equilibria. As in Proposition 1, the accommodating initial stock is evaluated at each allocation; this comparison does not hold a common exogenous initial money quantity fixed across different initial output ratios. For each stochastic member, $q_t < \bar{q} < 1$ and $\widehat{\pi}(q_t) < 0$, so it remains separated from the fully stabilized allocation. □

C.2 Proof of Proposition 3 (the inflation-indexed rule)

Fix a shape pair satisfying (A.2) and (A.6). On the operating range $\underline{\widehat{\pi}} < \pi_t - \bar{\pi} < \widehat{\pi}(\bar{q})$, the rule is

$$\tilde{m}_t^{\mathrm{s}} = \tilde{d}_t Y_t^n \left(\frac{\pi_t - \bar{\pi} - \underline{\widehat{\pi}}}{\beta}\right)^{1/\zeta}, \qquad \sigma_t^{m/d} \geq 0, \tag{C.2}$$

where $\sigma_t^{m/d}$ is the loading of $d\log(\tilde{m}_t^{\mathrm{s}}/\tilde{d}_t)$ on dZ_t. Assume $\tilde{m}_0 = \tilde{d}_0 Y_0^n q_0$ with $q_0 \in (0, \bar{q})$.

Proof. Combining (C.2) with money clearing gives

$$q_t = \left(\frac{\widehat{\pi}_t - \underline{\widehat{\pi}}}{\beta}\right)^{1/\zeta} \in (0, \bar{q}), \qquad \widehat{\pi}_t = \widehat{\pi}(q_t).$$

At $t = 0$, the same identity gives the prescribed q_0. No Markov restriction on candidate loadings is needed. Lemma A.3 requires $\sigma_t^\pi = \beta\zeta q_t^\zeta \sigma_t^s$ and the volatility quadratic. Since $F(q_t) > 0$, its two roots of (A.7) are

$$\begin{aligned} \sigma^s(q) &= \frac{-\sigma + \sqrt{\sigma^2 + 2(1+\zeta)F(q)}}{1+\zeta} > 0, \\ \sigma_-^s(q) &= \frac{-\sigma - \sqrt{\sigma^2 + 2(1+\zeta)F(q)}}{1+\zeta} < -\frac{2\sigma}{1+\zeta}. \end{aligned} \tag{C.3}$$

Money clearing implies $\log(\tilde{m}_t^{\mathrm{s}}/\tilde{d}_t) = \log Y_t^n + \log q_t$, whose Brownian loading is $\sigma + \sigma_t^s$. The negative root therefore gives

$$\sigma + \sigma_-^s(q_t) < -\frac{\sigma(1-\zeta)}{1+\zeta} < 0,$$

which violates the loading restriction. Under this root, output, and with it real money relative to its demand coefficient, would fall after a positive productivity innovation. The positive root gives $\sigma + \sigma^s(q_t) > 0$. It follows that every equilibrium has $\sigma_t^s = \sigma^s(q_t)$ almost everywhere and solves (A.9). Global pathwise uniqueness from Proposition A.1 then identifies its state path with the constructed solution from q_0, and every other equilibrium process follows from the graph, policy rule and clearing. This proves uniqueness.

Conversely, take that global interior solution and the inflation and loading processes used in the passive-supply proof. Lemma A.3 verifies the two limiting equations. The inverse graph in (C.2) equals q_t, so supply equals $\tilde{d}_t Y_t^n q_t$, clears the market, and matches the given initial stock. The loading of $\log(\tilde{m}_t^s/\tilde{d}_t)$ is $\sigma + \sigma^s(q_t) > 0$. The policy rate follows the rule (2.20), and the price level and allocations are well defined as above. Hence the money rule selects this solution as the unique limiting equilibrium. □

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